# VANISHING GEODESIC DISTANCE ON SPACES OF SUBMANIFOLDS AND DIFFEOMORPHISMS

PETER W. MICHOR, DAVID MUMFORD

Abstract. The $L^2$-metric or Fubini-Study metric on the non-linear Grassmannian of all submanifolds of type $M$ in a Riemannian manifold $(N, g)$ induces geodesic distance 0. We discuss another metric which involves the mean curvature and shows that its geodesic distance is a good topological metric. The vanishing phenomenon for the geodesic distance holds also for all diffeomorphism groups for the $L^2$-metric.

## 1. Introduction

In [10] we studied the $L^2$-Riemannian metric on the space of all immersions $S^1 \to \mathbb{R}^2$. This metric is invariant under the group $\operatorname{Diff}(S^1)$ and we found that it induces *vanishing geodesic distance* on the quotient space $\operatorname{Imm}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$. In this paper we extend this result to the general situation $\operatorname{Imm}(M, N)/\operatorname{Diff}(M)$ for any compact manifold $M$ and Riemannian manifold $(N, g)$ with $\dim N > \dim M$. On the open subset $\operatorname{Emb}(M, N)/\operatorname{Diff}(M)$, which may be identified with the space of all submanifolds of diffeomorphism type $M$ in $N$ (the non-linear Grassmanian or differentiable 'Chow' variety) this says that the infinite dimensional analog of the Fubini Study metric induces vanishing geodesic distance. The picture that emerges for these infinite-dimensional manifolds is quite interesting: there are simple expressions for the Christoffel symbols and curvature tensor, the geodesic equations are simple and of hyperbolic type and, at least in the case of plane curves, the geodesic spray exists locally. *But* the curvature is positive and unbounded in some high frequency directions, so these spaces wrap up on themselves arbitrarily tightly, allowing the infimum of path lengths between two given points to be zero.

We also carry over to the general case the stronger metric from [10] which weights the $L^2$ metric using the second fundamental form. It turns out that we have only to use the mean curvature in order to get positive geodesic distances, hence a good topological metric on the space $\operatorname{Emb}(S^1, \mathbb{R}^2)/\operatorname{Diff}(S^1)$. The reason is that the first variation of the volume of a submanifold depends on the mean curvature and the key step is showing that the square root of the volume of $M$ is Lipschitz in our stronger metric. The formula for this metric is:

$$G_f^A(h, k) := \int_M (1 + A\| \operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}}) g(h, k) \operatorname{vol}(f^*g)$$

2000 *Mathematics Subject Classification.* Primary 58B20, 58D15, 58E12.
Both authors were supported by NSF Grant 007 4276. PWM was supported by FWF Projects P 14195 and P 17108 and by Centre Bernoulli, Lausanne.

where $S^f$ is the second fundamental form of the immersion $f$; section 3 contains the relevant estimates. In section 4 we also compute the sectional curvature of the $L^2$-metric in the hope to relate the vanishing of the geodesic distance to unbounded positivity of the sectional curvature: by going through ever more positively curved parts of the space we can find ever shorter curves between any two submanifolds.

In the final section 5 we show that the vanishing of the geodesic distance also occurs on the Lie group of all diffeomorphisms on each connected Riemannian manifold. Short paths between any 2 diffeomorphisms are constructed by using rapidly moving compression waves in which individual points are trapped for relatively long times. We compute the sectional curvature also in this case.

## 2. The manifold of immersions

**2.1. Conventions.** Let $M$ be a compact smooth connected manifold of dimension $m \geq 1$ and let $(N, g)$ be a connected Riemannian manifold of dimension $n > m$. We shall use the following spaces and manifolds of smooth mappings.

$\operatorname{Diff}(M)$, the regular Lie group ([8], 38.4) of all diffeomorphisms of $M$.

$\operatorname{Diff}_{x_0}(M)$, the subgroup of diffeomorphisms fixing $x_0 \in M$.

$\operatorname{Emb} = \operatorname{Emb}(M, N)$, the manifold of all smooth embeddings $M \to N$.

$\operatorname{Imm} = \operatorname{Imm}(M, N)$, the manifold of all smooth immersions $M \to N$. For an immersion $f$ the tangent space with foot point $f$ is given by $T_f \operatorname{Imm}(M, N) = C^\infty_f(M, TN) = \Gamma(f^*TN)$, the space of all vector fields along $f$.

$\operatorname{Imm}_f = \operatorname{Imm}_f(M, N)$, the manifold of all smooth *free* immersions $M \to N$, i.e., those with trivial isotropy group for the right action of $\operatorname{Diff}(M)$ on $\operatorname{Imm}(M, N)$.

$B_e = B_e(M, N) = \operatorname{Emb}(M, N)/\operatorname{Diff}(M)$, the manifold of submanifolds of type $M$ in $N$, the base of a smooth principal bundle, see **2.2**.

$B_i = B_i(M, N) = \operatorname{Imm}(M, N)/\operatorname{Diff}(M)$, an infinite dimensional 'orbifold', whose points are, roughly speaking, smooth immersed submanifolds of type $M$ in $N$, see **2.4**.

$B_{i,f} = B_{i,f}(M, N) = \operatorname{Imm}_f(M, \mathbb{R}^2)/\operatorname{Diff}(M)$, a manifold, the base of a principal fiber bundle, see **2.3**.

For a smooth curve $f : \mathbb{R} \to C^\infty(M, N)$ corresponding to a mapping $f : \mathbb{R} \times M \to N$, we shall denote by $Tf$ the curve of tangent mappings, so that $Tf(t)(X_x) = T_x(f(t, \quad)).X_x$. The time derivative will be denoted by either $\partial_t f = f_t : \mathbb{R} \times M \to TN$.

**2.2. The principal bundle of embeddings** $\operatorname{Emb}(M, N)$**.** We recall some basic results whose proof can be found in [8]):

*(A) The set* $\operatorname{Emb}(M, N)$ *of all smooth embeddings* $M \to N$ *is an open subset of the smooth Fréchet manifold* $C^\infty(M, N)$ *of all smooth mappings* $M \to N$ *with the*

*$C^\infty$-topology. It is the total space of a smooth principal bundle $\pi : \mathrm{Emb}(M,N) \to B_e(M,N)$ with structure group $\mathrm{Diff}(M)$, the smooth regular Lie group group of all diffeomorphisms of $M$, whose base $B_e(M,N)$ is the smooth Fréchet manifold of all submanifolds of $N$ of type $M$, i.e., the smooth manifold of all simple closed curves in $N$.* ([8], 44.1)

*(B) This principal bundle admits a smooth principal connection described by the horizontal bundle whose fiber $\mathcal{N}_c$ over $c$ consists of all vector fields $h$ along $f$ such that $g(h, Tf) = 0$. The parallel transport for this connection exists and is smooth.* ([8], 39.1 and 43.1)

**2.3**. **Free immersions.** The manifold $\mathrm{Imm}(M,N)$ of all immersions $M \to N$ is an open set in the manifold $C^\infty(M,N)$ and thus itself a smooth manifold. An immersion $f : M \to N$ is called *free* if $\mathrm{Diff}(M)$ acts freely on it, i.e., $f \circ \varphi = c$ for $\varphi \in \mathrm{Diff}(M)$ implies $\varphi = \mathrm{Id}$. We have the following results:

- *If $\varphi \in \mathrm{Diff}(M)$ has a fixed point and if $f \circ \varphi = f$ for some immersion $f$ then $\varphi = \mathrm{Id}$.* This is ([4], 1.3).
- *If for $f \in \mathrm{Imm}(M,N)$ there is a point $x \in f(M)$ with only one preimage then $f$ is a free immersion.* This is ([4], 1.4). There exist free immersions without such points.
- **The manifold** $B_{i,f}(M,N)$ ([4], 1.5) *The set $\mathrm{Imm}_f(M,N)$ of all free immersions is open in $C^\infty(M,N)$ and thus a smooth submanifold. The projection*
$$\pi : \mathrm{Imm}_f(M,N) \to \frac{\mathrm{Imm}_f(M,N)}{\mathrm{Diff}(M)} =: B_{i,f}(M,N)$$
*onto a Hausdorff smooth manifold is a smooth principal fibration with structure group $\mathrm{Diff}(M)$. By* ([8], 39.1 and 43.1) *this fibration admits a smooth principal connection described by the horizontal bundle with fiber $\mathcal{N}_c$ consisting of all vector fields $h$ along $f$ such that $g(h, Tf) = 0$. This connection admits a smooth parallel transport over each smooth curve in the base manifold.*

We might view $\mathrm{Imm}_f(M,N)$ as the nonlinear Stiefel manifold of parametrized submanifolds of type $M$ in $N$ and consequently $B_{i,f}(M,N)$ as the nonlinear Grassmannian of unparametrized submanifolds of type $M$.

**2.4**. **Non free immersions.** Any immersion is proper since $M$ is compact and thus by ([4], 2.1) the orbit space $B_i(M,N) = \mathrm{Imm}(M,N)/\mathrm{Diff}(M)$ is Hausdorff. Moreover, by ([4], 3.1 and 3.2) for any immersion $f$ the isotropy group $\mathrm{Diff}(M)_f$ is a finite group which acts as group of covering transformations for a finite covering $q_c : M \to \bar{M}$ such that $f$ factors over $q_c$ to a free immersion $\bar{f} : \bar{M} \to N$ with $\bar{f} \circ q_c = f$. Thus the subgroup $\mathrm{Diff}_{x_0}(M)$ of all diffeomorphisms $\varphi$ fixing $x_0 \in M$ acts freely on $\mathrm{Imm}(M,N)$. Moreover, for each $f \in \mathrm{Imm}$ the submanifold $\mathcal{Q}(f)$ from **4.4**, (1) is a slice in a strong sense:

- $\mathcal{Q}(f)$ is invariant under the isotropy group $\mathrm{Diff}(M)_f$.
- If $\mathcal{Q}(f) \circ \varphi \cap \mathcal{Q}(f) \neq \emptyset$ for $\varphi \in \mathrm{Diff}(M)$ then $\varphi$ is already in the isotropy group $\varphi \in \mathrm{Diff}(M)_f$.

- $\mathcal{Q}(f) \circ \operatorname{Diff}(M)$ is an invariant open neigbourhood of the orbit $f \circ \operatorname{Diff}(M)$ in $\operatorname{Imm}(M,N)$ which admits a smooth retraction $r$ onto the orbit. The fiber $r^{-1}(f \circ \varphi)$ equals $\mathcal{Q}(f \circ \varphi)$.

Note that also the action

$$\operatorname{Imm}(M,N) \times \operatorname{Diff}(M) \to \operatorname{Imm}(M,N) \times \operatorname{Imm}(M,N), \qquad (f,\varphi) \mapsto (f, f \circ \varphi)$$

is proper so that all assumptions and conclusions of Palais' slice theorem [13] hold. This results show that the orbit space $B_i(M,N)$ has only singularities of orbifold type times a Fréchet space. We may call the space $B_i(M,N)$ an infinite dimensional *orbifold*. The projection $\pi : \operatorname{Imm}(M,N) \to B_i(M,N) = \operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ is a submersion off the singular points and has only mild singularities at the singular strata. The normal bundle $\mathcal{N}_f$ mentioned in **2.2** is well defined and is a smooth vector subbundle of the tangent bundle. We do not have a principal bundle and thus no principal connections, but we can prove the main consequence, the existence of horizontal paths, directly:

**2.5. Proposition.** *For any smooth path $f$ in $\operatorname{Imm}(M,N)$ there exists a smooth path $\varphi$ in $\operatorname{Diff}(M)$ with $\varphi(t, \quad) = \operatorname{Id}_M$ depending smoothly on $f$ such that the path $h$ given by $h(t,\theta) = f(t,\varphi(t,\theta))$ is horizontal: $g(h_t, Th) = 0$.*

**Proof.** Let us write $h = f \circ \varphi$ for $h(t,x) = f(t,\varphi(t,x))$, etc. We look for $\varphi$ as the integral curve of a time dependent vector field $\xi(t,x)$ on $M$, given by $\partial_t \varphi = \xi \circ \varphi$. We want the following expression to vanish:

$$\begin{aligned} g\big(\partial_t(f\circ\varphi), T(f\circ\varphi)\big) &= g\big((\partial_t f \circ \varphi + (Tf\circ\varphi).\partial_t\varphi, (Tf\circ\varphi).T\varphi\big) \\ &= \big(g(\partial_t f, Tf)\circ\varphi\big).T\varphi + g\big((Tf\circ\varphi)(\xi\circ\varphi), (Tf\circ\varphi).T\varphi\big) \\ &= \big(\big(g(\partial_t f, Tf) + g(Tf.\xi, Tf)\big)\circ\varphi\big).T\varphi \end{aligned}$$

Since $T\varphi$ is everywhere invertible we get

$$0 = g\big(\partial_t(f\circ\varphi), T(f\circ\varphi)\big) \iff 0 = g(\partial_t f, Tf) + g(Tf.\xi, Tf)$$

and the latter equation determines the non-autonomous vector field $\xi$ uniquely. $\square$

**2.6**. **Curvatures of an immersion.** Consider a fixed immersion $f \in \operatorname{Imm}(M,N)$. The *normal bundle* $N(f) = Tf^\perp \subset f^*TN \to M$ has fibers $N(f)_x = \{Y \in T_{f(x)}N : g(Y, T_x f.X) = 0 \text{ for all } X \in T_xM\}$. Every vector field $h : M \to TN$ along $f$ then splits as $h = Tf.h^\top + h^\perp$ into its tangential component $h^\top \in \mathfrak{X}(M)$ and its normal component $h^\perp \in \Gamma(N(f))$.

Let $\nabla^g$ be the Levi-Civita covariant derivative of $g$ on $N$ and let $\nabla^{f^*g}$ the Levi-Civita covariant derivative of the pullback metric $f^*g$ on $M$. The *shape operator* or *second fundamental form* $S^f \in \Gamma(S^2T^*M \otimes N(f))$ of $f$ is then given by

$$S^f(X,Y) = \nabla^g_X(Tf.Y) - Tf.\nabla^{f^*g}_X Y \quad \text{for } X,Y \in \mathfrak{X}(M). \tag{1}$$

It splits into the following irreducible components under the action of the group $O(T_xM)\times O(N(f)_x)$: the mean curvature $\operatorname{Tr}^{f^*g}(S^f) = \operatorname{Tr}((f^*g)^{-1}\circ S^f) \in \Gamma(N(f))$ and the trace free shape operator $S^f_0 = S^f - \operatorname{Tr}^{f^*g}(S^f)$. For $X \in \mathfrak{X}(M)$ and

$\xi \in \Gamma(N(f))$, i.e., a normal vector field along $f$, we may also split $\nabla^g_X \xi$ into the components which are tangential and normal to $Tf.TM$,

$$\nabla^g_X \xi = -Tf.L^f_\xi(X) + \nabla^{N(f)}_X \xi \tag{2}$$

where $\nabla^{N(f)}$ is the induced connection in the normal bundle respecting the metric $g^{N(f)}$ induced by $g$, and where the *Weingarten* tensor field $L^f \in \Gamma(N(f)^* \otimes T^*M \otimes TM)$ corresponds to the shape operator via the formula

$$(f^*g)(L^f_\xi(X), Y) = g^{N(f)}(S^f(X,Y), \xi). \tag{3}$$

Let us also split the Riemann curvature $R^g$ into tangential and normal parts: For $X_i \in \mathfrak{X}(M)$ or $T_xM$ we have (*theorema egregium*):

$$\begin{aligned} g(R^g(Tf.X_1, Tf.X_2)(Tf.X_3), Tf.X_4) &= (f^*g)(R^{f^*g}(X_1,X_2)X_3, X_4)+ \\ &+ g^{N(f)}(S^f(X_1,X_3), S^f(X_2,X_4)) - g^{N(f)}(S^f(X_2,X_3), S^f(X_1,X_4)). \end{aligned} \tag{4}$$

The normal part of $R^g$ is then given by (*Codazzi-Mainardi equation*):

$$\begin{aligned} &(R^g(Tf.X_1, Tf.X_2)(Tf.X_3))^\perp = \\ &= \big(\nabla^{N(f)\otimes T^*M\otimes T^*M}_{X_1} S^f\big)(X_2,X_3) - \big(\nabla^{N(f)\otimes T^*M\otimes T^*M}_{X_2} S^f\big)(X_1,X_3). \end{aligned} \tag{5}$$

**2.7**. **Volumes of an immersion.** For an immersion $f \in \mathrm{Imm}(M,N)$, we consider the volume density $\mathrm{vol}^g(f) = \mathrm{vol}(f^*g) \in \mathrm{Vol}(M)$ on $M$ given by the local formula $\mathrm{vol}^g(f)|_U = \sqrt{\det((f^*g)_{ij})}|du^1 \wedge \dots \wedge du^m|$ for any chart $(U, u : U \to \mathbb{R}^m)$ of $M$, and the induced volume function $\mathrm{Vol}^g : \mathrm{Imm}(M,N) \to \mathbb{R}_{>0}$ which is given by $\mathrm{Vol}^g(f) = \int_M \mathrm{vol}(f^*g)$. The tangent mapping of $\mathrm{vol} : \Gamma(S^2_{>0}T^*M) \to \mathrm{Vol}(M)$ is given by $d\,\mathrm{vol}(\gamma)(\eta) = \frac{1}{2}\mathrm{Tr}(\gamma^{-1}.\eta)\,\mathrm{vol}(\gamma)$. We consider the pullback mapping $P_g : f \mapsto f^*g, P_g : \mathrm{Imm}(M,N) \to \Gamma(S^2_{>0}T^*M)$. A version of the following lemma is [7], 1.6.

**Lemma.** *The derivative of* $\mathrm{vol}^g = \mathrm{vol} \circ P_g : \mathrm{Imm}(M,N) \to \mathrm{Vol}(M)$ *is*

$$\begin{aligned} d\,\mathrm{vol}^g(h) &= d(\mathrm{vol} \circ P_g)(h) \\ &= -\mathrm{Tr}^{f^*g}(g(S^f, h^\perp))\,\mathrm{vol}(f^*g) + \tfrac{1}{2}\mathrm{Tr}^{f^*g}(\mathcal{L}_{h^\top}(f^*g))\,\mathrm{vol}(f^*g). \\ &= -(g(\mathrm{Tr}^{f^*g}(S^f), h^\perp))\,\mathrm{vol}(f^*g) + \mathrm{div}^{f^g}(h^\top)\,\mathrm{vol}(f^*g). \end{aligned}$$

**Proof.** We consider a curve $t \mapsto f(t, \quad)$ in Imm with $\partial_t|_0 f = h$. We also use a chart $(U, u : U \to \mathbb{R}^m)$ on $M$. Then we have

$$f^*g|_U = \sum_{i,j} (f^*g)_{ij} du^i \otimes du^j = \sum_{i,j} g(Tf.\partial_{u_i}, Tf.\partial_{u^j}) du^i \otimes du^j$$

$$\partial_t \,\mathrm{vol}^g(f)|_U = \frac{\det((f^*g)_{ij})(f^*g)^{kl}\partial_t(f^*g)_{lk}}{2\sqrt{\det((f^*g)_{ij})}} |du^1 \wedge \dots \wedge du^m|$$

where

$$\begin{aligned} \partial_t(f^*g)_{ij} &= \partial_t g(Tf.\partial_{u^i}, Tf.\partial_{u^j}) \\ &= g(\nabla^g_{\partial_t}(Tf.\partial_{u^i}), Tf.\partial_{u^j}) + g(\partial_{u^i}, \nabla^g_{\partial_t}(Tf.\partial_{u^j})) \\ g(\nabla^g_{\partial_t} Tf.\partial_{u^i}, Tf.\partial_{u^j}) &= g(\nabla^g_{\partial_{u^i}} Tf.\partial_t + Tf.\,\mathrm{Tor} + Tf.[\partial_t, \partial_{u^i}], Tf.\partial_{u^j}) \\ &= g(\nabla^g_{\partial_{u^i}}(Tf.\partial_t)^\perp, Tf.\partial_{u^j}) + g(\nabla^g_{\partial_{u^i}}(Tf.\partial_t)^\top, Tf.\partial_{u^j}) \end{aligned}$$

$$\begin{aligned}
g(\nabla^g_{\partial_{u^i}}(\partial_t f)^\perp, Tf.\partial_{u^j}) &= g(-Tf.L_{(\partial_t f)^\perp}\partial_{u^i}, Tf.\partial_{u^j}) + g(\nabla^{N(f)}_{\partial_{u^i}}(\partial_t f)^\perp, Tf.\partial_{u^j}) \\
&= -(f^*g)(L_{(\partial_t f)^\perp}\partial_{u^i}, \partial_{u^j}) \\
&= -g(S^f(\partial_{u^i}, \partial_{u^j}), (\partial_t f)^\perp) \\
g(\nabla^g_{\partial_{u^i}}(\partial_t f)^\top, Tf.\partial_{u^j}) &= (f^*g)(\nabla^{f^*g}_{\partial_{u^i}}(\partial_t f)^\top, \partial_{u^j}) + 0 \\
&= (f^*g)(\nabla^{f^*g}_{(\partial_t f)^\top}\partial_{u^i} + \operatorname{Tor} - [(\partial_t f)^\top, \partial_{u^i}], \partial_{u^j}), \\
\partial_t (f^*g)_{ij} &= -2g(S^f(\partial_{u^i}, \partial_{u^j}), (\partial_t f)^\perp) + (\mathcal{L}_{(\partial_t f)^\top}(f^*g))(\partial_{u^i}, \partial_{u^j})
\end{aligned}$$

This proves the first formula. For the second one note that

$$\tfrac{1}{2}\operatorname{Tr}((f^*g)^{-1}\mathcal{L}_{h^\top}(f^*g))\operatorname{vol}(f^*g) = \mathcal{L}_{h^\top}(\operatorname{vol}(f^*g)) = \operatorname{div}^{f^*g}(h^\top)\operatorname{vol}(f^*g). \quad \square$$

## 3. Metrics on spaces of mappings

**3.1. The metric $G^A$.** Let $h, k \in C^\infty_f(M, TN)$ be two tangent vectors with foot point $f \in \operatorname{Imm}(M,N)$, i.e., vector fields along $f$. Let the induced volume density be $\operatorname{vol}(f^*g)$. We consider the following weak Riemannian metric on $\operatorname{Imm}(M,N)$, for a constant $A \geq 0$:

$$G^A_f(h,k) := \int_M \Big(1 + A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}}\Big)\, g(h,k)\operatorname{vol}(f^*g)$$

where $\operatorname{Tr}^{f^*g}(S^f) \in N(f)$ is the mean curvature, a section of the normal bundle, and $\|\operatorname{Tr}^{f^*g}(S^f)\|_{g^{N(f)}}$ is its norm. The metric $G^A$ is invariant for the action of $\operatorname{Diff}(M)$. This makes the map $\pi : \operatorname{Imm}(M,N) \to B_i(M,N)$ into a *Riemannian submersion* (off the singularities of $B_i(M,N)$).

Now we can determine the bundle $\mathcal{N} \to \operatorname{Imm}(M,N)$ of tangent vectors which are normal to the $\operatorname{Diff}(M)$-orbits. The tangent vectors to the orbits are $T_f(f \circ \operatorname{Diff}(M)) = \{Tf.\xi : \xi \in \mathfrak{X}(M)\}$. Inserting this for $k$ into the expression of the metric $G$ we see that

$$\begin{aligned}
\mathcal{N}_f &= \{h \in C^\infty(M,TN) : g(h, Tf) = 0\} \\
&= \Gamma(N(f)),
\end{aligned}$$

the space of sections of the normal bundle. This is independent of $A$.

A tangent vector $h \in T_f \operatorname{Imm}(M,N) = C^\infty_f(M,TN) = \Gamma(f^*TN)$ has an orthonormal decomposition

$$h = h^\top + h^\perp \in T_f(f \circ \operatorname{Diff}^+(M)) \oplus \mathcal{N}_f$$

into smooth tangential and normal components.

Since the Riemannian metric $G^A$ on $\operatorname{Imm}(M,N)$ is invariant under the action of $\operatorname{Diff}(M)$ it induces a metric on the quotient $B_i(M,N)$ as follows. For any $F_0, F_1 \in B_i$, consider all liftings $f_0, f_1 \in \operatorname{Imm}$ such that $\pi(f_0) = F_0, \pi(f_1) = F_1$ and all smooth curves $t \mapsto f(t, \quad)$ in $\operatorname{Imm}(M,N)$ with $f(0,\cdot) = f_0$ and $f(1,\cdot) = f_1$. Since the metric $G^A$ is invariant under the action of $\operatorname{Diff}(M)$ the arc-length of the curve $t \mapsto \pi(f(t,\cdot))$ in $B_i(M,N)$ is given by

$$L^{\text{hor}}_{G^A}(f) := L_{G^A}(\pi(f(t,\cdot))) =$$

$$= \int_0^1 \sqrt{G^A_{\pi(f)}(T_f\pi.f_t, T_f\pi.f_t)}\,dt = \int_0^1 \sqrt{G^A_f(f_t^\perp, f_t^\perp)}\,dt =$$
$$= \int_0^1 \Big(\int_M (1 + A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}}) g(f_t^\perp, f_t^\perp)\operatorname{vol}(f^*g)\Big)^{\frac{1}{2}} dt$$

In fact the last computation only makes sense on $B_{i,f}(M,N)$ but we take it as a motivation. The metric on $B_i(M,N)$ is defined by taking the infimum of this over all paths $f$ (and all lifts $f_0, f_1$):

$$\operatorname{dist}^{B_i}_{G^A}(F_1,F_2) = \inf_f L^{\text{hor}}_{G^A}(f).$$

**3.2. Theorem.** *Let $A = 0$. For $f_0, f_1 \in \operatorname{Imm}(M,N)$ there exists always a path $t \mapsto f(t,\cdot)$ in $\operatorname{Imm}(M,N)$ with $f(0,\cdot) = f_0$ and $\pi(f(1,\cdot)) = \pi(f_1)$ such that $L^{hor}_{G^0}(f)$ is arbitrarily small.*

**Proof.** Take a path $f(t,\theta)$ in $\operatorname{Imm}(M,N)$ from $f_0$ to $f_1$ and make it horizontal using **2.4** so that that $g(f_t, Tf) = 0$; this forces a reparametrization on $f_1$.

Let $\alpha : M \to [0,1]$ be a surjective Morse function whose singular values are all contained in the set $\{\frac{k}{2N} : 0 \le k \le 2N\}$ for some integer $N$. We shall use integers $n$ below and we shall use only multiples of $N$.

Then the level sets $M_r := \{x \in M : \alpha(x) = r\}$ are of Lebesque measure 0. We shall also need the slices $M_{r_1,r_2} := \{x \in M : r_1 \le \alpha(x) \le r_2\}$. Since $M$ is compact there exists a constant $C$ such that the following estimate holds uniformly in $t$:

$$\int_{M_{r_1,r_2}} \operatorname{vol}(f(t,\quad)^*g) \le C(r_2 - r_1)\int_M \operatorname{vol}(f(t,\quad)^*g)$$

Let $\tilde f(t,x) = f(\varphi(t,\alpha(x)),x)$ where $\varphi : [0,1]\times[0,1] \to [0,1]$ is given as in [10], 3.10 (which also contains a figure illustrating the construction) by

$$\varphi(t,\alpha) = \begin{cases} 2t(2n\alpha - 2k) & \text{for } 0 \le t \le 1/2,\ \frac{2k}{2n} \le \alpha \le \frac{2k+1}{2n} \\ 2t(2k+2-2n\alpha) & \text{for } 0 \le t \le 1/2,\ \frac{2k+1}{2n} \le \alpha \le \frac{2k+2}{2n} \\ 2t-1+2(1-t)(2n\alpha-2k) & \text{for } 1/2 \le t \le 1,\ \frac{2k}{2n} \le \alpha \le \frac{2k+1}{2n} \\ 2t-1+2(1-t)(2k+2-2n\alpha) & \text{for } 1/2 \le t \le 1,\ \frac{2k+1}{2n} \le \alpha \le \frac{2k+2}{2n}. \end{cases}$$

Then we get $T\tilde f = \varphi_\alpha.d\alpha.f_t + Tf$ and $\tilde f_t = \varphi_t.f_t$ where

$$\varphi_\alpha = \begin{cases} +4nt \\ -4nt \\ +4n(1-t) \\ -4n(1-t) \end{cases}, \qquad \varphi_t = \begin{cases} 4n\alpha - 4k \\ 4k+4-4n\alpha \\ 2-4n\alpha+4k \\ -(2-4n\alpha+4k) \end{cases}.$$

We use horizontality $g(f_t, Tf) = 0$ to determine $\tilde f_t^\perp = \tilde f_t + T\tilde f(X)$ where $X \in TM$ satisfies $0 = g(\tilde f_t + T\tilde f(X), T\tilde f(\xi))$ for all $\xi \in TM$. We also use

$$d\alpha(\xi) = f^*g(\operatorname{grad}^{f^*g}\alpha, \xi) = g(Tf(\operatorname{grad}^{f^*g}\alpha), Tf(\xi))$$

and get

$$0 = g(\tilde f_t + T\tilde f(X), T\tilde f(\xi))$$

$$
\begin{aligned}
&= g\Big(\varphi_t f_t + \varphi_\alpha d\alpha(X) f_t + Tf(X), \varphi_\alpha d\alpha(\xi) f_t + Tf(\xi)\Big)\\
&= \varphi_t.\varphi_\alpha.(f^*g)(\operatorname{grad}^{f^*g}\alpha,\xi)\|f_t\|_g^2 +\\
&\quad + \varphi_\alpha^2.(f^*g)(\operatorname{grad}^{f^*g}\alpha, X).(f^*g)(\operatorname{grad}^{f^*g}\alpha,\xi)\|f_t\|_g^2 + g(Tf(X),Tf(\xi))\\
&= (\varphi_t.\varphi_\alpha + \varphi_\alpha^2.(f^*g)(\operatorname{grad}^{f^*g}\alpha, X))\|f_t\|_g^2 (f^*g)(\operatorname{grad}^{f^*g}\alpha,\xi) + (f^*g)(X,\xi)
\end{aligned}
$$

This implies that $X = \lambda \operatorname{grad}^{f^*g}\alpha$ for a function $\lambda$ and in fact we get

$$
\tilde f_t^\perp = \frac{\varphi_t}{1+\varphi_\alpha^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2} f_t - \frac{\varphi_t\varphi_\alpha\|f_t\|_g^2}{1+\varphi_\alpha^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2} Tf(\operatorname{grad}^{f^*g}\alpha)
$$

and

$$
\|\tilde f_t\|_g^2 = \frac{\varphi_t^2\|f_t\|_g^2}{1+\varphi_\alpha^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2}
$$

From $T\tilde f = \varphi_\alpha.d\alpha.f_t + Tf$ and $g(f_t, Tf) = 0$ we get for the volume form

$$
\operatorname{vol}(\tilde f^*g) = \sqrt{1+\varphi_\alpha^2\,\|d\alpha\|_{f^*g}^2\|f_t\|_g^2}\,\operatorname{vol}(f^*g).
$$

For the horizontal length we get

$$
\begin{aligned}
&L^{\mathrm{hor}}(\tilde f) = \int_0^1 \Big(\int_M \|\tilde f_t^\perp\|_g^2 \operatorname{vol}(\tilde f^*g)\Big)^{\frac12} dt =\\
&= \int_0^1 \Big(\int_M \frac{\varphi_t^2\|f_t\|_g^2}{\sqrt{1+\varphi_\alpha^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2}} \operatorname{vol}(f^*g)\Big)^{\frac12} dt =\\
&= \int_0^{\frac12}\Bigg(\sum_{k=0}^{n-1}\Big(\int_{M_{\frac{2k}{2n},\frac{2k+1}{2n}}} \frac{(4n\alpha-4k)^2\|f_t\|_g^2}{\sqrt{1+(4nt)^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2}}\operatorname{vol}(f^*g) +\\
&\qquad\qquad + \int_{M_{\frac{2k+1}{2n},\frac{2k+2}{2n}}} \frac{(4k+4-4n\alpha)^2\|f_t\|_g^2}{\sqrt{1+(4nt)^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2}}\operatorname{vol}(f^*g)\Big)\Bigg)^{\frac12} dt +\\
&\quad + \int_{\frac12}^1\Bigg(\sum_{k=0}^{n-1}\Big(\int_{M_{\frac{2k}{2n},\frac{2k+1}{2n}}} \frac{(2-4n\alpha+4k)^2\|f_t\|_g^2}{\sqrt{1+(4n(1-t))^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2}}\operatorname{vol}(f^*g) +\\
&\qquad\qquad + \int_{M_{\frac{2k+1}{2n},\frac{2k+2}{2n}}} \frac{(2-4n\alpha+4k)^2\|f_t\|_g^2}{\sqrt{1+(4n(1-t))^2\|d\alpha\|_{f^*g}^2\|f_t\|_g^2}}\operatorname{vol}(f^*g)\Big)\Bigg)^{\frac12} dt
\end{aligned}
$$

Let $\varepsilon > 0$. The function $(t,x)\mapsto \|f_t(\varphi(t,\alpha(x)),x)\|_g^2$ is uniformly bounded. On $M_{\frac{2k}{2n},\frac{2k+1}{2n}}$ the function $4n\alpha - 4k$ has values in $[0,2]$. Choose disjoint geodesic balls centered at the finitely many singular values of the Morse function $\alpha$ of total $f^*g$-volume $< \varepsilon$. Restricted to the union $M_{\mathrm{sing}}$ of these balls the integral above is $O(1)\varepsilon$. So we have to estimate the integrals on the complement $\tilde M = M \setminus M_{\mathrm{sing}}$ where the function $\|d\alpha\|_{f^*g}$ is uniformly bounded from below by $\eta > 0$.

Let us estimate one of the sums above. We use the fact that the singular points of the Morse function $\alpha$ lie all on the boundaries of the sets $\tilde M_{\frac{2k}{2n},\frac{2k+1}{2n}}$ so that we

can transform the integrals as follows:

$$\sum_{k=0}^{n-1}\int_{\tilde M_{\frac{2k}{2n},\frac{2k+1}{2n}}}\frac{(4n\alpha-4k)^2\|f_t\|_g^2}{\sqrt{1+(4nt)^2\|d\alpha\|^2_{f^*g}\|f_t\|_g^2}}\operatorname{vol}(f^*g)=$$
$$=\sum_{k=0}^{n-1}\int_{\frac{2k}{2n}}^{\frac{2k+1}{2n}}\int_{\tilde M_r}\frac{(4nr-4k)^2\|f_t\|_g^2}{\sqrt{1+(4nt)^2\|d\alpha\|^2_{f^*g}\|f_t\|_g^2}}\frac{\operatorname{vol}(i_r^*f^*g)}{\|d\alpha\|_{f^*g}}\,dr$$

We estimate this sum of integrals: Consider first the set of all $(t,r,x\in M_r)$ such that $|f_t(\varphi(t,r),x)|<\varepsilon$. There we estimate by

$$O(1).n.16n^2.\varepsilon^2.(r^3/3)|_{r=0}^{r=1/2n}=O(\varepsilon).$$

On the complementary set where $|f_t(\varphi(t,r),x)|\ge\varepsilon$ we estimate by

$$O(1).n.16n^2.\frac{1}{4nt\eta^2\varepsilon}(r^3/3)|_{r=0}^{r=1/2n}=O(\frac{1}{nt\eta^2\varepsilon})$$

which goes to 0 if $n$ is large enough. The other sums of integrals can be estimated similarly, thus $L^{\text{hor}}(\tilde f)$ goes to 0 for $n\to\infty$. It is clear that one can approximate $\varphi$ by a smooth function whithout changing the estimates essentially. $\square$

**3.3**. **A Lipschitz bound for the volume in $G^A$.** We apply the Cauchy-Schwarz inequality to the derivative **2.7** of the volume $\operatorname{Vol}^g(f)$ along a curve $t\mapsto f(t,\quad)\in\operatorname{Imm}(M,N)$:

$$\begin{aligned}
\partial_t\operatorname{Vol}^g(f)&=\partial_t\int_M\operatorname{vol}^g(f(t,\quad))=\int_M d\operatorname{vol}^g(f)(\partial_t f)\\
&=-\int_M\operatorname{Tr}^{f^*g}(g(S^f,f_t^\perp))\operatorname{vol}(f^*g)\le\Big|\int_M\operatorname{Tr}^{f^*g}(g(S^f,f_t^\perp))\operatorname{vol}(f^*g)\Big|\\
&\le\Big(\int_M 1^2\operatorname{vol}(f^*g)\Big)^{\frac12}\Big(\int_M\operatorname{Tr}^{f^*g}(g(S^f,f_t^\perp))^2\operatorname{vol}(f^*g)\Big)^{\frac12}\\
&\le\operatorname{Vol}^g(f)^{\frac12}\frac{1}{\sqrt A}\Big(\int_M(1+A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}})\,g(f_t^\perp,f_t^\perp)\operatorname{vol}(f^*g)\Big)^{\frac12}
\end{aligned}$$

Thus

$$\begin{aligned}
\partial_t(\sqrt{\operatorname{Vol}^g(f)})&=\frac{\partial_t\operatorname{Vol}^g(f)}{2\sqrt{\operatorname{Vol}^g(f)}}\le\\
&\le\frac{1}{2\sqrt A}\Big(\int_M(1+A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}})\,g(f_t^\perp,f_t^\perp)\operatorname{vol}(f^*g)\Big)^{\frac12}
\end{aligned}$$

and by using (**3.1**) we get

$$\begin{aligned}
\sqrt{\operatorname{Vol}^g(f_1)}-\sqrt{\operatorname{Vol}^g(f_0)}&=\int_0^1\partial_t(\sqrt{\operatorname{Vol}^g(f)})\,dt\\
&\le\frac{1}{2\sqrt A}\int_0^1\Big(\int_M(1+A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}})\,g(f_t^\perp,f_t^\perp)\operatorname{vol}(f^*g)\Big)^{\frac12}dt\\
&=\frac{1}{2\sqrt A}L^{\text{hor}}_{G^A}(f).
\end{aligned}$$

If we take the infimum over all curves connecting $f_0$ with the $\operatorname{Diff}(M)$-orbit through $f_1$ we get:

**Proposition. Lipschitz continuity of $\sqrt{\operatorname{Vol}^g} : B_i(M,N) \to \mathbb{R}_{\geq 0}$.** *For $F_0$ and $F_1$ in $B_i(M,N) = \operatorname{Imm}(M,N)/\operatorname{Diff}(M)$ we have for $A > 0$:*

$$\sqrt{\operatorname{Vol}^g(F_1)} - \sqrt{\operatorname{Vol}^g(F_0)} \leq \frac{1}{2\sqrt{A}} \operatorname{dist}^{B_i}_{G^A}(F_1, F_2).$$

**3.4. Bounding the area swept by a path in $B_i$.** We want to bound the area swept out by a curve starting from $F_0$ to any immersed submanifold $F_1$ nearby in our metric. First we use the Cauchy-Schwarz inequality in the Hilbert space $L^2(M, \operatorname{vol}(f(t,\quad)^*g))$ to get

$$\begin{aligned}\int_M 1.\|f_t\|_g \operatorname{vol}(f^*g) &= \langle 1, \|f_t\|_g\rangle_{L^2} \leq \\ &\leq \|1\|_{L^2}\|c_t\|_{L^2} = \Big(\int_M \operatorname{vol}(f^*g)\Big)^{\frac12}\Big(\int_M |f_t| \operatorname{vol}(f^*g)\Big)^{\frac12}.\end{aligned}$$

Now we assume that the variation $f(t,x)$ is horizontal, so that $g(f_t, Tf) = 0$. Then $L_{G^A}(f) = L^{\text{hor}}_{G^A}(f)$. We use this inequality and then the intermediate value theorem of integral calculus to obtain

$$\begin{aligned}L^{\text{hor}}_{G^A}(f) = L_{G^A}(f) &= \int_0^1 \sqrt{G^A_f(f_t, f_t)}\,dt \\ &= \int_0^1 \Big(\int_M (1 + A\|\operatorname{Tr}^{f^*}(S^f)\|^2_{f^*g})\|f_t\|^2 \operatorname{vol}(f^*g)\Big)^{\frac12} dt \\ &\geq \int_0^1 \Big(\int_M \|f_t\|^2 \operatorname{vol}(f^*g)\Big)^{\frac12} dt \\ &\geq \int_0^1 \Big(\int_M \operatorname{vol}(f(t,\quad)^*g)\Big)^{-\frac12} \int_M \|f_t(t,\quad)\|_g \operatorname{vol}(f(t,\quad)^*g)\,dt \\ &= \Big(\int_M \operatorname{vol}(f(t_0,\quad)^*g)\Big)^{-\frac12} \int_0^1 \int_M \|f_t(t,\quad)\|_g \operatorname{vol}(f(t,\quad)^*g)\,dt \\ &\qquad \text{for some intermediate value } 0 \leq t_0 \leq 1, \\ &\geq \frac{1}{\sqrt{\operatorname{Vol}^g(f(t_0,\quad))}} \int_{[0,1]\times M} \operatorname{vol}^{m+1}(f^*g)\end{aligned}$$

**Proposition. Area swept out bound.** *If $f$ is any path from $F_0$ to $F_1$, then*

$$\begin{pmatrix}(m+1) - \textit{volume of the region swept} \\ \textit{out by the variation } f\end{pmatrix} \leq \max_t \sqrt{\operatorname{Vol}^g(f(t,\quad))} \cdot L^{hor}_{G^A}(f).$$

Together with the Lipschitz continuity **3.3** this shows that the geodesic distance $\inf L^{B_i}_{G^A}$ separates points, at least in the base space $B(M,N)$ of embeddings.

**3.5. Horizontal energy of a path as anisotropic volume.** We consider a path $t \mapsto f(t,\quad)$ in $\operatorname{Imm}(M,N)$. It projects to a path $\pi \circ f$ in $B_i$ whose energy is:

$$E_{G^A}(\pi \circ f) = \tfrac12 \int_a^b G^A_{\pi(f)}(T\pi.f_t, T\pi.f_t)\,dt = \tfrac12 \int_a^b G^A_f(f_t^\perp, f_t^\perp)\,dt =$$

$$= \tfrac{1}{2}\int_a^b \int_M (1 + A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}})g(f_t^\perp, f_t^\perp)\operatorname{vol}(f^*g)\,dt.$$

We now consider the graph $\gamma_f : [a,b]\times M \ni (t,x) \mapsto (t, f(t,x)) \in [a,b]\times N$ of the path $f$ and its image $\Gamma_f$, an immersed submanifold with boundary of $\mathbb{R}\times N$. We want to describe the horizontal energy as a functional on the space of immersed submanifolds with fixed boundary, remembering the fibration of $\mathrm{pr}_1 : \mathbb{R}\times N \to \mathbb{R}$. We get:

$$\begin{aligned} &E_{G^A}(\pi\circ f) = \\ &= \tfrac{1}{2}\int_{[a,b]\times M}\Big(1 + A\|\operatorname{Tr}^{f^*g}(S^f)\|^2_{g^{N(f)}}\Big)\frac{\|f_t^\perp\|^2}{\sqrt{1+\|f_t^\perp\|^2_g}}\operatorname{vol}(\gamma_f^*(dt^2+g)) \end{aligned}$$

Now $\|f_t^\perp\|_g$ depends only on the graph $\Gamma_f$ and on the fibration over time, since any reparameterization of $\Gamma_f$ which respects the fibration over time is of the form $(t,x)\mapsto (t, f(t,\varphi(t,x)))$ for some path $\varphi$ in $\operatorname{Diff}(M)$ starting at the identity, and $(\partial_t|_0 f(t,\varphi(t,x)))^\perp = f_t^\perp$. So the above expression is intrinsic for the graph $\Gamma_f$ and the fibration. In order to find a geodesic from the shape $\pi(f(a,\ \ ))$ to the shape $\pi(f(b,\ \ ))$ one has to find an immersed surface which is a critical point for the functional $E_{G^A}$ above. This is a Plateau-problem with anisotropic volume.

## 4. The geodesic equation and the curvature on $B_i$

**4.1. The geodesic equation of $G^0$ in $\operatorname{Imm}(M,N)$.** The energy of a curve $t\mapsto f(t,\ \ )$ in $\operatorname{Imm}(M,N)$ for $G^0$ is

$$E_{G^0}(f) = \tfrac{1}{2}\int_a^b\int_M g(f_t,f_t)\operatorname{vol}(f^*g).$$

**The geodesic equation for $G^0$**

$$\boxed{\begin{aligned}&\nabla^g_{\partial_t} f_t + \operatorname{div}^{f^*g}(f_t^\top)f_t - g(f_t^\perp, \operatorname{Tr}^{f^*g}(S^f))f_t + \\ &\qquad + \tfrac{1}{2}Tf.\operatorname{grad}^{f^*g}(\|f_t\|^2_g) + \tfrac{1}{2}\|f_t\|^2_g\operatorname{Tr}^{f^*g}(S^f) = 0\end{aligned}} \tag{1}$$

**Proof.** A different proof is in [7], 2.2. For a function $a$ on $M$ we shall use

$$\begin{aligned}\int_M a\operatorname{div}^{f^*g}(X)\operatorname{vol}(f^*g) &= \int_M a\mathcal{L}_X(\operatorname{vol}(f^*g)) \\ &= \int_M \mathcal{L}_X(a\operatorname{vol}(f^*g)) - \int_M \mathcal{L}_X(a)\operatorname{vol}(f^*g) \\ &= -\int_M (f^*g)(\operatorname{grad}^{f^*g}(a), X)\operatorname{vol}(f^*g)\end{aligned}$$

in calculating the first variation of the energy with fixed ends:

$$\begin{aligned}\partial_s E_{G^0}(f) &= \tfrac{1}{2}\int_a^b\int_M\Big(\partial_s g(f_t,f_t)\ \operatorname{vol}(f^*g) + g(f_t,f_t)\,\partial_s\operatorname{vol}(f^*g)\Big)dt \\ &= \int_a^b\int_M\Big(g(\nabla^g_{\partial_s}f_t, f_t)\ \operatorname{vol}(f^*g) + \tfrac{1}{2}\|f_t\|^2_g\operatorname{div}^{f^*g}(f_s^\top)\operatorname{vol}(f^*g) \\ &\qquad\qquad - \tfrac{1}{2}\|f_t\|^2_g g(f_s^\perp, \operatorname{Tr}^{f^*g}(S^f))\operatorname{vol}(f^*g)\Big)dt\end{aligned}$$

For the first summand we have:

$$\begin{aligned}
&\int_a^b \int_M g(\nabla^g_{\partial_s} f_t, f_t)\ \mathrm{vol}(f^*g)\,dt = \int_a^b \int_M g(\nabla^g_{\partial_t} f_s, f_t)\ \mathrm{vol}(f^*g)\,dt\\
&= \int_a^b \int_M (\partial_t g(f_s, f_t) - g(f_s, \nabla^g_{\partial_t} f_t))\,\mathrm{vol}(f^*g)\,dt\\
&= -\int_a^b \int_M g(f_s, f_t)\partial_t \,\mathrm{vol}(f^*g)\,dt - \int_a^b \int_M g(f_s, \nabla^g_{\partial_t} f_t)\,\mathrm{vol}(f^*g)\,dt\\
&= \int_a^b \int_M \Big(-g(f_s, f_t)\,\mathrm{div}^{f^*g}(f_t^\top) + g(f_s, f_t) g(f_t^\perp, \mathrm{Tr}^{f^*g}(S^f)) -\\
&\qquad\qquad - g(f_s, \nabla^g_{\partial_t} f_t)\Big)\,\mathrm{vol}(f^*g)\,dt
\end{aligned}$$

The second summand yields:

$$\begin{aligned}
&\int_a^b \int_M \tfrac12 \|f_t\|^2_g\,\mathrm{div}^{f^*g}(f_s^\top)\,\mathrm{vol}(f^*g)\,dt\\
&\qquad = -\int_a^b \int_M \tfrac12 (f^*g)(f_s^\top, \mathrm{grad}^{f^*g}(\|f_t\|^2_g))\,\mathrm{vol}(f^*g)\,dt\\
&\qquad = -\int_a^b \int_M \tfrac12 g(f_s, Tf.\,\mathrm{grad}^{f^*g}(\|f_t\|^2_g))\,\mathrm{vol}(f^*g)\,dt
\end{aligned}$$

Thus the first variation $\partial_s E_{G^0}(f)$ is:

$$\begin{aligned}
&\int_a^b \int_M g\Big(f_s, -\nabla^g_{\partial_t} f_t - \mathrm{div}^{f^*g}(f_t^\top) f_t + g(f_t^\perp, \mathrm{Tr}^{f^*g}(S^f)) f_t\\
&\qquad - \tfrac12 Tf.\,\mathrm{grad}^{f^*g}(\|f_t\|^2_g) - \tfrac12 \|f_t\|^2_g g(f_s^\perp, \mathrm{Tr}^{f^*g}(S^f))\Big)\,\mathrm{vol}(f^*g)\,dt \quad \square
\end{aligned}$$

**4.2. Geodesics for $G^0$ in $B_i(M,N)$.** We restrict to geodesics $t \mapsto f(t,\ \ )$ in $\mathrm{Imm}(M,N)$ which are horizontal: $g(f_t, Tf) = 0$. Then $f_t^\top = 0$ and $f_t = f_t^\perp$, so equation **4.1**.(1) becomes

$$\nabla^g_{\partial_t} f_t - g(f_t, \mathrm{Tr}^{f^*g}(S^f)) f_t + \tfrac12 Tf.\,\mathrm{grad}^{f^*g}(\|f_t\|^2_g) + \tfrac12 \|f_t\|^2_g \,\mathrm{Tr}^{f^*g}(S^f) = 0.$$

It splits into a vertical (tangential) part

$$-Tf.(\nabla_{\partial_t} f_t)^\top + \tfrac12 Tf.\,\mathrm{grad}^{f^*g}(\|f_t\|^2_g) = 0$$

which vanishes identically since

$$\begin{aligned}
(f^*g)(\mathrm{grad}^{f^*g}(\|f_t\|^2_g), X) &= X(g(f_t, f_t)) = 2g(\nabla_X f_t, f_t) = 2g(\nabla^g_{\partial_t} Tf.X, f_t)\\
&= 2\partial_t g(Tf.X, f_t) - 2g(Tf.X, \nabla^g_{\partial_t} f_t) = -2g(Tf.X, \nabla^g_{\partial_t} f_t),
\end{aligned}$$

and a horizontal (normal) part which is the geodesic equation in $B_i$:

$$(1)\qquad \boxed{\nabla^{N(f)}_{\partial_t} f_t - g(f_t, \mathrm{Tr}^{f^*g}(S^f)) f_t + \tfrac12 \|f_t\|^2_g \,\mathrm{Tr}^{f^*g}(S^f) = 0, \quad g(Tf, f_t) = 0}$$

**4.3. The induced metric of $G^0$ in $B_i(M,N)$ in a chart.** Let $f_0 : M \to N$ be a fixed immersion which will be the 'center' of our chart. Let $N(f_0) \subset f_0^*TN$ be the normal bundle to $f_0$. Let $\exp^g : N(f_0) \to N$ be the exponential map for the metric $g$ and let $V \subset N(f_0)$ be a neighborhood of the 0 section on which the exponential map is an immersion. Consider the mapping

$$\begin{aligned} \psi = \psi_{f_0} &: \Gamma(V) \to \mathrm{Imm}(M,N), \quad , \psi(\Gamma(V)) =: \mathcal{Q}(f_0), \\ \psi(a)(x) &= \exp^g(a(x)) = \exp^g_{f_0(x)}(a(x)). \end{aligned} \tag{1}$$

The inverse (on its image) of $\pi \circ \psi_f : \Gamma(V) \to B_i(M,N)$ is a smooth chart on $B_i(M,N)$. Our goal is to calculate the induced metric on this chart, that is

$$((\pi \circ \psi_{f_0})^* G^0_a)(b_1, b_2)$$

for any $a \in \Gamma(V), b_1, b_2 \in \Gamma(N(f_0))$. This will enable us to calculate the sectional curvatures of $B_i$.

We shall fix the section $a$ and work with the ray of points $t.a$ in this chart. Everything will revolve around the map:

$$f(t,x) = \psi(t.a)(x) = \exp^g(t.a(x)).$$

We shall also use a fixed chart $(M \supset U \xrightarrow{u} \mathbb{R}^m)$ on $M$ with $\partial_i = \partial/\partial u^i$. Then $x \mapsto (t \mapsto f(t,x)) = \exp^g_{f_0(x)}(t.a(x))$ is a variation consisting entirely of geodesics, thus:

$$\begin{aligned} &t \mapsto \partial_i f(t,x) = Tf.\partial_i =: Z_i(t,x,a) \textit{ is the Jacobi field along } t \mapsto f(t,x) \textit{ with} \\ &Z_i(0,x,a) = \partial_i|_x \exp_{f_0(x)}(0) = \partial_i|_x f_0 = T_x f_0.\partial_i|_x, \\ &(\nabla^g_{\partial_t} Z_i)(0,x) = (\nabla^g_{\partial_t} Tf.\partial_i)(0,x) = (\nabla^g_{\partial_i} Tf.\partial_t)(0,x) = \\ &\qquad = \nabla^g_{\partial_i}(\partial_t|_0 \exp_{f_0(x)}(t.a(x))) = (\nabla^g_{\partial_i} a)(x). \end{aligned} \tag{2}$$

Then the pullback metric is given by

$$\begin{aligned} f^*g = \psi(ta)^*g = g(Tf,Tf) &= \sum_{i,j=1}^m g(Tf.\partial_i, Tf, \partial_j)\, du^i \otimes du^j \\ &= \sum_{i,j=1}^m g(Z_i, Z_j)\, du^i \otimes du^j. \end{aligned} \tag{3}$$

The induced volume density is:

$$\mathrm{vol}(f^*g) = \sqrt{\det(g(Z_i,Z_j))}|du^1 \wedge \dots \wedge du^m| \tag{4}$$

Moreover we have for $a \in \Gamma(V)$ and $b \in \Gamma(N(f_0))$

$$\begin{aligned} (T_{ta}\psi.tb)(x) &= \partial_s|_0 \exp^g_{f_0(x)}(ta(x) + stb(x))) \\ &= Y(t,x,a,b) \quad \text{ for the Jacobi field } Y \text{ along } t \mapsto f(t,x) \text{ with} \\ Y(0,x,a,b) &= 0_{f_0(x)}, \\ (\nabla^g_{\partial_t} Y(\quad, x,a,b))(0) &= \nabla^g_{\partial_t}\partial_s \exp^g_{f_0(x)}(ta(x) + stb(x)) \\ &= \nabla^g_{\partial_s}\partial_t|_0 \exp^g_{f_0(x)}(ta(x)+stb(x)) \\ &= \nabla^g_{\partial_s}(a(x) + s.b(x))|_{s=0} = b(x). \end{aligned} \tag{5}$$

Now we want to split $T_a\psi.b$ into vertical (tangential) and horizontal parts with respect to the immersion $\psi(ta) = f(t, \quad)$. The tangential part has locally the form

$$\begin{aligned}
Tf.(T_{ta}\psi.tb)^\top &= \sum_{i=1}^m c^i\, Tf.\partial_i = \sum_{i=1}^m c^i\, Z_i \quad \text{where for all } j\\
g(Y, Z_j) &= g\left(\sum_{i=1}^m c^i\, Z_i, Z_j\right) = \sum_{i=1}^m c^i\, (f^*g)_{ij},\\
c^i &= \sum_{j=1}^m (f^*g)^{ij} g(Y, Z_j).
\end{aligned}$$

Thus the horizontal part is

$$(T_{ta}\psi.tb)^\perp = Y^\perp = Y - \sum_{i=1}^m c^i\, Z_i = Y - \sum_{i,j=1}^m (f^*g)^{ij}\, g(Y, Z_j)\, Z_i \tag{6}$$

Thus the induced metric on $B_i(M,N)$ has the following expression in the chart $(\pi\circ\psi_{f_0})^{-1}$, where $a \in \Gamma(V)$ and $b_1, b_2 \in \Gamma(N(f_0))$:

$$\begin{aligned}
((\pi\circ\psi_{f_0})^*G^0)_{ta}(b_1, b_2) &= G^0_{\pi(\psi(ta))}(T_{ta}(\pi\circ\psi)b_1, T_{ta}(\pi\circ\psi)b_2)\\
&= G^0_{\psi(ta)}((T_{ta}\psi.b_1)^\perp, (T_{ta}\psi.b_2)^\perp)\\
&= \int_M g((T_{ta}\psi.b_1)^\perp, (T_{ta}\psi.b_2)^\perp)\ \mathrm{vol}(f^*g)\\
&= \int_M \frac{1}{t^2} g\Big(Y(b_1) - \sum_{i,j}(f^*g)^{ij}\, g(Y(b_1), Z_j)\, Z_i\ ,\ Y(b_2)\Big) \qquad (7)\\
&\qquad\qquad \sqrt{\det(g(Z_i, Z_j))}|du^1\wedge\dots\wedge du^m|
\end{aligned}$$

**4.4. Expansion to order 2 of the induced metric of $G^0$ in $B_i(M,N)$ in a chart.** We use the setting of **4.3**, the Einstein summation convention, and the abbreviations $f_i := \partial_i f_0 = \partial_{u^i} f_0$ and $\nabla^g_i := \nabla^g_{\partial_i} = \nabla^g_{\partial_{u^i}}$. We compute the expansion in $t$ up to order 2 of the metric **4.3**.(7). Our method is to use the Jacobi equation

$$\nabla^g_{\partial_t}\nabla^g_{\partial_t} Y = R^g(\dot c, Y)\dot c$$

which holds for any Jacobi field $Y$ along a geodesic $c$. By **2.6**.(2) we have:

$$g(\nabla^g_i a, f_j) = -(f_0^*g)(L_a^{f_0}(\partial_i), \partial_j) = -g(a, S^{f_0}(f_i, f_j)) = g(a, S^{f_0}_{ij}) \tag{1}$$

We start by expanding the pullback metric **4.3**.(3) and its inverse:

$$\begin{aligned}
\partial_t(f^*g)_{ij} &= \partial_t g(Z_i, Z_j) = g(\nabla^g_{\partial_t} Z_i, Z_j) + g(Z_i, \nabla^g_{\partial_t} Z_j)\\
\partial_t^2 g(Z_i, Z_j) &= g(\nabla^g_{\partial_t}\nabla^g_{\partial_t} Z_i, Z_j) + 2g(\nabla^g_{\partial_t} Z_i, \nabla^g_{\partial_t} Z_j) + g(Z_i, \nabla^g_{\partial_t}\nabla^g_{\partial_t} Z_j)\\
(f^*g)_{ij} &= (f_0^*g)_{ij} + t\big(g(\nabla^g_i a, f_j) + g(f_i, \nabla^g_j a)\big) +\\
&\quad + \tfrac12 t^2\big(g(R^g(a, f_i)a, f_j) + 2g(\nabla^g_i a, \nabla^g_j a) + g(f_i, R^g(a, f_j)a)\big)\\
&\quad + O(t^3)\\
&= (f_0^*g)_{ij} - 2t(f_0^*g)(L_a^{f_0}(\partial_i), \partial_j)\\
&\quad + t^2\big(g(R^g(a, f_i)a, f_j) + g(\nabla^g_i a, \nabla^g_j a)\big) + O(t^3) \qquad (2)
\end{aligned}$$

We expand now the volume form $\mathrm{vol}(f^*g) = \sqrt{\det(g(Z_i,Z_j))}|du^1\wedge\cdots\wedge du^m|$. The time derivative at 0 of the inverse of the pullback metric is:

$$\partial_t(f^*g)^{ij}|_0 = -(f_0^*g)^{ik}(\partial_t|_0(f^*g)_{kl})(f_0^*g)^{lj} = -(f_0^*g)^{ik}(f_0^*g)(a,S^{f_0}_{kl})(f_0^*g)^{lj}$$

Therefore,

$$\begin{aligned}
\partial_t\sqrt{\det(g(Z_i,Z_j))} &= \tfrac12(f^*g)^{ij}\partial_t(g(Z_i,Z_j))\sqrt{\det(g(Z_i,Z_j))}\\
\partial_t^2\sqrt{\det(g(Z_i,Z_j))} &= \tfrac12\partial_t(f^*g)^{ij}\partial_t(g(Z_i,Z_j))\sqrt{\det(g(Z_i,Z_j))}\\
&\quad+\tfrac12(f^*g)^{ij}\partial_t^2(g(Z_i,Z_j))\sqrt{\det(g(Z_i,Z_j))}\\
&\quad+\tfrac12(f^*g)^{ij}\partial_t(g(Z_i,Z_j))\partial_t\sqrt{\det(g(Z_i,Z_j))},
\end{aligned}$$

and

$$\begin{aligned}
\mathrm{vol}(f^*g) &= \sqrt{\det(g(Z_i,Z_j))}|du^1\wedge\cdots\wedge du^m|\\
&= \Big(1-t\,\mathrm{Tr}(L_a^{f_0}) + t^2\Big(-\mathrm{Tr}(L_a^{f_0}\circ L_a^{f_0}) + \tfrac12(\mathrm{Tr}(L_a^{f_0}))^2\\
&\qquad+\tfrac12(f^*g)^{ij}\big(g(R^g(a,f_i)a,f_j)+g(\nabla_i^g a,\nabla_j^g a)\big)\Big)+O(t^3)\Big)\,\mathrm{vol}(f_0^*g) \qquad (3)
\end{aligned}$$

Moreover, by **2.6**.(2) we may split $\nabla_i^g a = -Tf_0.L_a^{f_0}(\partial_i) + \nabla_i^{N(f_0)}a$ and we write $\nabla_i^\perp a$ for $\nabla_i^{N(f_0)}a$ shortly. Thus:

$$\begin{aligned}
(f_0^*g)^{ij}g(\nabla_i^g a,\nabla_j^g a) &= (f_0^*g)^{ij}\,\big(g(Tf_0.L_a^{f_0}(\partial_i),Tf_0.L_a^{f_0}(\partial_j)) + g(\nabla_i^\perp a,\nabla_j^\perp a)\big)\\
&= \mathrm{Tr}(L_a^{f_0}\circ L_a^{f_0}) + (f_0^*g)^{ij}g(\nabla_i^\perp a,\nabla_j^\perp a)
\end{aligned}$$

and so that the tangential term above combines with the first $t^2$ term in the expansion of the volume, changing its coefficient from $-1$ to $-\frac12$.

Let us now expand

$$\begin{aligned}
g((T_{ta}\psi.tb_1)^\perp,(T_{ta}\psi.tb_2)^\perp) &= g\big(Y(b_1)-(f^*g)^{ij}g(Y(b_1),Z_j)\,Z_i\ ,\ Y(b_2)\big)\\
&= g(Y(b_1),Y(b_2)) - (f^*g)^{ij}g(Y(b_1),Z_j)g(Z_i,Y(b_2)).
\end{aligned}$$

We have:

$$\begin{aligned}
\partial_t g(Y(b_1),Y(b_2)) &= g(\nabla^g_{\partial_t}Y(b_1),Y(b_2)) + g(Y(b_1),\nabla^g_{\partial_t}Y(b_2))\\
\partial_t^2 g(Y(b_1),Y(b_2)) &= g(\nabla^g_{\partial_t}\nabla^g_{\partial_t}Y(b_1),Y(b_2)) + 2g(\nabla^g_{\partial_t}Y(b_1),\nabla^g_{\partial_t}Y(b_2))\\
&\quad+g(Y(b_1),\nabla^g_{\partial_t}\nabla^g_{\partial_t}Y(b_2))\\
&= 2g(R^g(a,Y(b_1))a,Y(b_2)) + 2g(\nabla^g_{\partial_t}Y(b_1),\nabla^g_{\partial_t}Y(b_2))\\
\partial_t g(Y(b_1),Z_j) &= g(\nabla^g_{\partial_t}Y(b_1),Z_j) + g(Y(b_1),\nabla^g_{\partial_t}Z_j)\\
\partial_t^2 g(Y(b_1),Z_j) &= 2g(R^g(a,Y(b_1))a,Z_j) + 2g(\nabla^g_{\partial_t}Y(b_1),\nabla^g_{\partial_t}Z_j)
\end{aligned}$$

Note that:

$$\begin{aligned}
&Y(0,h)=0,\quad (\nabla^g_{\partial_t}Y(h))(0)=h,\quad (\nabla^g_{\partial_t}\nabla^g_{\partial_t}Y(h))(0)=R^g(a,Y(0,h))a=0,\\
&(\nabla^g_{\partial_t}\nabla^g_{\partial_t}\nabla^g_{\partial_t}Y(h))(0)=R^g(a,\nabla^g_{\partial_t}Y(h)(0))a=R^g(a,h)a.
\end{aligned}$$

Thus:

$$g(Y(b_1),Y(b_2)) - (f^*g)^{ij}g(Y(b_1),Z_j)g(Z_i,Y(b_2))$$

$$= t^2 g(b_1,b_2) + t^4\big(\tfrac13 g(R^g(a,b_1)a,b_2) - (f_0^*g)^{ij} g(b_1,\nabla_j^\perp a) g(\nabla_i^\perp a, b_2)\big) + O(t^5).$$

The expansion of $G^0$ up to order 2 is thus:

$$\begin{aligned}
&((\pi\circ\psi_{f_0})^*G^0)_{ta}(b_1,b_2) =\\
&\quad= \int_M \frac{1}{t^2} g\Big(Y(b_1) - \sum_{i,j}(f^*g)^{ij}\, g(Y(b_1),Z_j)\, Z_i\ ,\ Y(b_2)\Big)\ \operatorname{vol}(f^*g)\\
&\quad= \int_M \Big(g(b_1,b_2)\operatorname{vol}(f_0^*g) - t\int_M g(b_1,b_2)\operatorname{Tr}(L_a^{f_0})\operatorname{vol}(f_0^*g)\\
&\qquad + t^2 \int_M \bigg(g(b_1,b_2)\Big(-\tfrac12 \operatorname{Tr}(L_a^{f_0}\circ L_a^{f_0}) + \tfrac12\operatorname{Tr}(L_a^{f_0})^2\\
&\qquad\qquad + \tfrac12 (f^*g)^{ij} g(R^g(a,f_i)a,f_j) + \tfrac12 (f^*g)^{ij} g(\nabla_i^\perp a, \nabla_j^\perp a)\Big)\\
&\qquad\qquad + \tfrac13 g(R^g(a,b_1)a,b_2) - (f_0^*g)^{ij} g(b_1,\nabla_j^\perp a) g(\nabla_i^\perp a, b_2)\bigg)\operatorname{vol}(f_0^*g)\\
&\qquad + O(t^3)
\end{aligned}\tag{4}$$

**4.5. Computation of the sectional curvature in $B_i(M,N)$ at $f_0$.** We use the following formula which is valid in a chart:

$$\begin{aligned}
2R_a(m,h,m,h) &= 2G^0_a(R_a(m,h)m,h) =\\
&= -2d^2G^0(a)(m,h)(h,m) + d^2G^0(a)(m,m)(h,h) + d^2G^0(a)(h,h)(m,m)\\
&\quad - 2G^0(\Gamma(h,m),\Gamma(m,h)) + 2G^0(\Gamma(m,m),\Gamma(h,h))
\end{aligned}$$

The sectional curvature at the two-dimensional subspace $P_a(m,h)$ of the tangent space which is spanned by $m$ and $h$ is then given by:

$$k_a(P(m,h)) = -\frac{G^0_a(R(m,h)m,h)}{\|m\|^2\|h\|^2 - G^0_a(m,h)^2}.$$

We compute this directly for $a=0$. From the expansion up to order 2 of $G^0_{ta}(b_1,b_2)$ in **4.4**.(4) we get

$$dG^0(0)(a)(b_1,b_2) = -\int_M g(b_1,b_2) g(a, \operatorname{Tr}^{f_0^*g} S^{f_0})\operatorname{vol}(f_0^*g)$$

and compute the Christoffel symbol:

$$\begin{aligned}
-2G^0_0(\Gamma_0(a,b),c) &= -dG^0(0)(c)(a,b) + dG^0(0)(a)(b,c) + dG^0(0)(b)(c,a)\\
&= \int_M \Big(g(a,b)g(c,\operatorname{Tr}^{f_0^*g}(S^{f_0})) - g(b,c)g(a,\operatorname{Tr}^{f_0^*g}(S^{f_0}))\\
&\qquad - g(c,a)g(b,\operatorname{Tr}^{f_0^*g}(S^{f_0}))\Big)\operatorname{vol}(f_0^*g)\\
&= \int_M g\Big(c,\ g(a,b)\operatorname{Tr}^{f_0^*g}(S^{f_0}) - \operatorname{Tr}(L_a^{f_0})b - \operatorname{Tr}(L_b^{f_0})a\Big)\operatorname{vol}(f_0^*g)\\
\Gamma_0(a,b) &= -\tfrac12 g(a,b)\operatorname{Tr}^{f_0^*g}(S^{f_0}) + \tfrac12\operatorname{Tr}(L_a^{f_0})b + \tfrac12\operatorname{Tr}(L_b^{f_0})a
\end{aligned}$$

The expansion **4.4**.(4) also gives:

$$\begin{aligned}
&\tfrac{1}{2!} d^2G^0_0(a_1,a_2)(b_1,b_2) =\\
&= \int_M \bigg(g(b_1,b_2)\Big(-\tfrac12\operatorname{Tr}(L_{a_1}^{f_0}\circ L_{a_2}^{f_0}) + \tfrac12\operatorname{Tr}(L_{a_1}^{f_0})\operatorname{Tr}(L_{a_2}^{f_0})
\end{aligned}$$

$$
\begin{aligned}
&\qquad + \tfrac{1}{2}(f^*g)^{ij} g(R^g(a_1,f_i)a_2,f_j) + \tfrac{1}{2}(f^*g)^{ij} g(\nabla_i^\perp a_1, \nabla_j^\perp a_2)\Big)\\
&+ \tfrac{1}{6} g(R^g(a_1,b_1)a_2,b_2) + \tfrac{1}{6} g(R^g(a_2,b_1)a_1,b_2)\\
&- \tfrac{1}{2}(f_0^*g)^{ij} g(b_1,\nabla_j^\perp a_1) g(\nabla_i^\perp a_2, b_2)\\
&- \tfrac{1}{2}(f_0^*g)^{ij} g(b_1,\nabla_j^\perp a_2) g(\nabla_i^\perp a_1, b_2)\Big) \operatorname{vol}(f_0^*g) + O(t^3)
\end{aligned}
$$

Thus we have:

$$
\begin{aligned}
&- d^2G^0(0)(x,y)(y,x) + \tfrac{1}{2} d^2G^0(0)(x,x)(y,y) + \tfrac{1}{2} d^2G^0(0)(y,y)(x,x) =\\
&= \int_M \Big( -2g(y,x)\Big( -\tfrac{1}{2}\operatorname{Tr}(L_x^{f_0}\circ L_y^{f_0}) + \tfrac{1}{2}\operatorname{Tr}(L_x^{f_0})\operatorname{Tr}(L_y^{f_0})\\
&\qquad\qquad + \tfrac{1}{2}(f_0^*g)^{ij} g(R^g(x,f_i)y,f_j) + \tfrac{1}{2}(f_0^*g)^{ij} g(\nabla_i^\perp x, \nabla_j^\perp y)\Big)\\
&\qquad + g(y,y)\Big( -\tfrac{1}{2}\operatorname{Tr}(L_x^{f_0}\circ L_x^{f_0}) + \tfrac{1}{2}\operatorname{Tr}(L_x^{f_0})\operatorname{Tr}(L_x^{f_0})\\
&\qquad\qquad + \tfrac{1}{2}(f_0^*g)^{ij} g(R^g(x,f_i)x,f_j) + \tfrac{1}{2}(f_0^*g)^{ij} g(\nabla_i^\perp x, \nabla_j^\perp x)\Big)\\
&\qquad + g(x,x)\Big( -\tfrac{1}{2}\operatorname{Tr}(L_y^{f_0}\circ L_y^{f_0}) + \tfrac{1}{2}\operatorname{Tr}(L_y^{f_0})\operatorname{Tr}(L_y^{f_0})\\
&\qquad\qquad + \tfrac{1}{2}(f_0^*g)^{ij} g(R^g(y,f_i)y,f_j) + \tfrac{1}{2}(f_0^*g)^{ij} g(\nabla_i^\perp y, \nabla_j^\perp y)\Big)\\
&\qquad + g(R^g(y,x)y,x)\\
&\qquad + (f_0^*g)^{ij}\left( g(y,\nabla_j^\perp y) g(\nabla_i^\perp x, x) + g(y,\nabla_j^\perp x) g(\nabla_i^\perp y, x)\right)\\
&\qquad - (f_0^*g)^{ij}\left( g(x,\nabla_j^\perp y) g(\nabla_i^\perp y, x) + g(y,\nabla_j^\perp x) g(\nabla_i^\perp x, y)\right)\Big) \operatorname{vol}(f_0^*g)
\end{aligned}
$$

For the second part of the curvature we have

$$
\begin{aligned}
&- G_0(\Gamma_0(x,y),\Gamma_0(y,x)) + G_0(\Gamma_0(y,y),\Gamma_0(x,x)) =\\
&= \tfrac{1}{4}\int_M \Big( (\|x\|_g^2\|y\|_g^2 - g(x,y)^2)\|\operatorname{Tr}^{f_0^*g}(S^f)\|_g^2 - 3\|\operatorname{Tr}(L_x^{f_0})y - \operatorname{Tr}(L_y^{f_0})x\|_g^2\Big) \operatorname{vol}(f_0^*g)
\end{aligned}
$$

To organize all these terms in the curvature tensor, note that they belong to *three* types: terms which involve the second fundamental form $L^{f_0}$, terms which involve the curvature tensor $R^g$ of $N$ and terms which involve the normal component of the covariant derivative $\nabla^\perp a$. There are 3 of the first type, two of the second and the ones of the third can be organized neatly into two also. The final curvature tensor is the integral over $M$ of their sum. Here are the terms in detail:

(1) *Terms involving the trace of products of $L$'s.* These are:

$$
-\tfrac{1}{2}\Big( g(y,y)\operatorname{Tr}(L_x^{f_0}\circ L_x^{f_0}) - 2g(x,y)\operatorname{Tr}(L_x^{f_0}\circ L_y^{f_0}) + g(x,x)\operatorname{Tr}(L_y^{f_0}\circ L_y^{f_0})\Big).
$$

Note that $x$ and $y$ are sections of the normal bundle $N(f_0)$, so we may define $x\wedge y$ to be the induced section of $\bigwedge^2 N(f_0)$. Then the expression inside the parentheses is a positive semi-definite quadratic function of $x\wedge y$. To see this, note a simple linear algebra fact – that if $Q(a,b)$ is any positive semi-definite inner product on $\mathbb{R}^n$, then

$$
\widetilde{Q}(a\wedge b, c\wedge d) =
$$

$$< a,c > Q(b,d) - < a,d > Q(b,c) + < b,d > Q(a,c) - < b,c > Q(a,d)$$

$$\widetilde{Q}(a\wedge b, a\wedge b) = \|a\|^2 Q(b,b) - 2 < a,b > Q(a,b) + \|b\|^2 Q(a,a)$$

is a positive semi-definite inner product on $\bigwedge^2 V$. In particular, $\operatorname{Tr}(L^{f_0}_x \circ L^{f_0}_y)$ is a positive semi-definite inner product on the normal bundle, hence it defines a positive semi-definite inner product $\widetilde{\operatorname{Tr}}(L^{f_0}\circ L^{f_0})$ on $\bigwedge^2 N(f_0)$. Thus:

$$\operatorname{term}(1) = -\tfrac12 \widetilde{\operatorname{Tr}}(L^{f_0}\circ L^{f_0})(x\wedge y) \le 0.$$

(2) *Terms involving trace of one $L$.* We have terms both from the second and first derivatives of $G$, namely:

$$\tfrac12\Big(g(y,y)\operatorname{Tr}(L^{f_0}_x)^2 - 2g(x,y)\operatorname{Tr}(L^{f_0}_x)\operatorname{Tr}(L^{f_0}_y) + g(x,x)\operatorname{Tr}(L^{f_0}_y)^2\Big).$$

and

$$-\tfrac34\|\operatorname{Tr}(L^{f_0}_x)y - \operatorname{Tr}(L^{f_0}_y)x\|^2_g$$

which are the same up to their coefficients. Their sum is:

$$\operatorname{term}(2) = -\tfrac14\|\operatorname{Tr}(L^{f_0}_x)y - \operatorname{Tr}(L^{f_0}_y)x\|^2_g \le 0.$$

Note that this is a function of $x\wedge y$ also.

(3) *The term involving the norm of the second fundamental form.* Since $\|x\|^2_g\|y\|^2_g - g(x,y)^2 = \|x\wedge y\|^2_g$, this term is just:

$$\operatorname{term}(3) = +\tfrac14\|x\wedge y\|^2_g\|\operatorname{Tr}^g(S^{f_0})\|^2_g \ge 0.$$

(4) *The curvature of $N$ term.* This is:

$$\operatorname{term}(4) = g(R^g(x,y)x,y).$$

Note that because of the skew-symmetry of the Riemann tensor, this is a function of $x\wedge y$ also.

(5) *The Ricci-curvature-like term.* The other curvature terms are:

$$\tfrac12(f_0^*g)^{ij}\Big(g(x,x)g(R^g(y,f_i)y,f_j) - 2g(x,y)g(R^g(x,f_i)y,f_j) + g(y,y)g(R^g(x,f_i)x,f_j)\Big)$$

If $V$ and $W$ are two perpendicular subspaces of the tangent space $TN_p$ at a point $p$, then we can define a 'cross Ricci curvature' $\operatorname{Ric}(V,W)$ in terms of bases $\{v_i\},\{w_j\}$ of $V$ and $W$ by:

$$\operatorname{Ric}(V,W) = g^{ij}g^{kl}g(R^g(v_i,w_k)v_j,w_l).$$

Then this term factors as:

$$\operatorname{term}(5) = \|x\wedge y\|^2_g\operatorname{Ric}(TM,\operatorname{span}(x,y)).$$

(6-7) *Terms involving the covariant derivative of $a$.* It is remarkable that, so far, every term in the curvature tensor of $B_i$ vanishes if $x\wedge y\equiv 0$, e.g., if the codimension of $N$ in $M$ is one! Now we have the terms:

$$\begin{aligned}(f_0^*g)^{ij}\Big(&g(x,y)g(\nabla^\perp_i x,\nabla^\perp_j y) - \tfrac12 g(x,x)g(\nabla^\perp_i y,\nabla^\perp_j y) - \tfrac12 g(y,y)g(\nabla^\perp_i x,\nabla^\perp_j x)\\ &- g(x,\nabla^\perp_i x)g(y,\nabla^\perp_j y) - g(x,\nabla^\perp_i y)g(y,\nabla^\perp_j x)\\ &+ g(x,\nabla^\perp_i y)g(x,\nabla^\perp_j y) + g(y,\nabla^\perp_i x)g(y,\nabla^\perp_j x)\Big).\end{aligned}$$

To understand this expression, we need a linear algebra computation, namely that if $a, b, a', b' \in \mathbb{R}^n$, then:

$$\begin{aligned}&< a,b >< a',b' > - < a,a' >< b,b' > - < a,b' >< b,a' > -\\ &\quad -\tfrac{1}{2} < a,a >< b',b' > -\tfrac{1}{2} < b,b >< a',a' > + < a,b' >^2 + < b,a' >^2 =\\ &\qquad = \tfrac{1}{2}(< a,b' > - < b,a' >)^2 - \tfrac{1}{2}\|a \wedge b' - b \wedge a'\|^2\end{aligned}$$

Note that the term $g(x, \nabla^\perp y)$ (without an $i$) is a section of $\Omega^1_M$ and the sum over $i$ and $j$ is just the norm in $\Omega^1_M$, so the above computation applies and the expression splits into 2 terms:

$$\begin{aligned}\text{term}(6) &= -\tfrac{1}{2}\|(g(x, \nabla^\perp y) - g(y, \nabla^\perp x)\|^2_{\Omega^1_M} \le 0\\ \text{term}(7) &= \tfrac{1}{2}\|x \wedge \nabla^\perp y - y \wedge \nabla^\perp x\|^2_{\Omega^1_M \otimes \wedge^2 N(f)} \ge 0.\end{aligned}$$

Altogether, we get that the Riemann curvature of $B_i$ is the integral over $M$ of the sum of the above 7 terms. We have the Corollary:

**Corollary.** *If the codimension of $M$ in $N$ is one, then all sectional curvatures of $B_i$ are non-negative. For any codimension, sectional curvature in the plane spanned by $x$ and $y$ is non-negative if $x$ and $y$ are parallel, i.e., $x \wedge y = 0$ in $\bigwedge^2 T^*N$.*

In general, the negative terms in the curvature tensor (giving positive sectional curvature) are clearly connected with the vanishing of geodesic distance: in some directions the space wraps up on itself in tighter and tighter ways. However, in codimension two or more with a flat ambient space $N$ (so terms (4) and (5) vanish), there seem to exist conflicting tendencies making $B_i$ close up or open up: terms (1), (2) and (6) give positive curvature, while terms (3) and (7) give negative curvature. It would be interesting to explore the geometrical meaning of these, e.g., for manifolds of space curves.

## 5. Vanishing geodesic distance on groups of diffeomorphisms

**5.1. The $H^0$-metric on groups of diffeomorphisms.** Let $(N, g)$ be a smooth connected Riemannian manifold, and let $\mathrm{Diff}_c(N)$ be the group of all diffeomorphisms with compact support on $N$, and let $\mathrm{Diff}_0(N)$ be the subgroup of those which are diffeotopic in $\mathrm{Diff}_c(N)$ to the identity; this is the connected component of the identity in $\mathrm{Diff}_c(N)$, which is a regular Lie group in the sense of [8], section 38, see [8], section 42. The Lie algebra is $\mathfrak{X}_c(N)$, the space of all smooth vector fields with compact support on $N$, with the negative of the usual bracket of vector fields as Lie bracket. Moreover, $\mathrm{Diff}_0(N)$ is a simple group (has no nontrivial normal subgroups), see [5], [14], [9]. The *right invariant* $H^0$-metric on $\mathrm{Diff}_0(N)$ is then given as follows, where $h, k : N \to TN$ are vector fields with compact support along $\varphi$ and where $X = h \circ \varphi^{-1}, Y = k \circ \varphi^{-1} \in \mathfrak{X}_c(N)$:

$$\begin{aligned}G^0_\varphi(h,k) &= \int_N g(h,k)\,\mathrm{vol}(\varphi^* g) = \int_N g(X \circ \varphi, Y \circ \varphi)\varphi^*\,\mathrm{vol}(g)\\ &= \int_N g(X,Y)\,\mathrm{vol}(g) \qquad (1)\end{aligned}$$

**5.2. Theorem.** *Geodesic distance on* $\mathrm{Diff}_0(N)$ *with respect to the* $H^0$*-metric vanishes.*

**Proof.** Let $[0,1] \ni t \mapsto \varphi(t,\quad)$ be a smooth curve in $\mathrm{Diff}_0(N)$ between $\varphi_0$ and $\varphi_1$. Consider the curve $u = \varphi_t \circ \varphi^{-1}$ in $\mathfrak{X}_c(N)$, the right logarithmic derivative. Then for the length and the energy we have:

$$L_{G^0}(\varphi) = \int_0^1 \sqrt{\int_N \|u\|_g^2 \operatorname{vol}(\mathrm{g})}\, dt \tag{1}$$

$$E_{G^0}(\varphi) = \int_0^1 \int_N \|u\|_g^2 \operatorname{vol}(\mathrm{g})\; dt \tag{2}$$

$$L_{G^0}(\varphi)^2 \le E_{G^0}(\varphi) \tag{3}$$

(4) Let us denote by $\mathrm{Diff}_0(N)^{E=0}$ the set of all diffeomorphisms $\varphi \in \mathrm{Diff}_0(N)$ with the following property: For each $\varepsilon > 0$ there exists a smooth curve from the identity to $\varphi$ in $\mathrm{Diff}_0(N)$ with energy $\le \varepsilon$.

(5) *We claim that* $\mathrm{Diff}_0(N)^{E=0}$ *coincides with the set of all diffeomorphisms which can by reached from the identity by a smooth curve of arbitraily short* $G^0$*-length.* This follows by (3).

(6) *We claim that* $\mathrm{Diff}_0(N)^{E=0}$ *is a normal subgroup of* $\mathrm{Diff}_0(N)$. Let $\varphi_1 \in \mathrm{Diff}_0(N)^{E=0}$ and $\psi \in \mathrm{Diff}_0(N)$. For any smooth curve $t \mapsto \varphi(t,\quad)$ from the identity to $\varphi_1$ with energy $E_{G^0}(\varphi) < \varepsilon$ we have

$$\begin{aligned}
E_{G^0}(\psi^{-1}\circ\varphi\circ\psi) &= \int_0^1 \int_N \|T\psi^{-1}\circ\varphi_t\circ\psi\|_g^2 \operatorname{vol}((\psi^{-1}\circ\varphi\circ\psi)^*g) \\
&\le \sup_{x\in N} \|T_x\psi^{-1}\|^2 \cdot \int_0^1 \int_N \|\varphi_t\circ\psi\|_g^2 (\varphi\circ\psi)^* \operatorname{vol}((\psi^{-1})^*g) \\
&\le \sup_{x\in N} \|T_x\psi^{-1}\|^2 \cdot \sup_{x\in N} \frac{\operatorname{vol}((\psi^{-1})^*g)}{\operatorname{vol}(g)} \cdot \int_0^1 \int_N \|\varphi_t\circ\psi\|_g^2\, (\varphi\circ\psi)^* \operatorname{vol}(g) \\
&\le \sup_{x\in N} \|T_x\psi^{-1}\|^2 \cdot \sup_{x\in N} \frac{\operatorname{vol}((\psi^{-1})^*g)}{\operatorname{vol}(g)} \cdot E_{G^0}(\varphi).
\end{aligned}$$

Since $\psi$ is a diffeomorphism with compact support, the two suprema are bounded. Thus $\psi^{-1}\circ\varphi_1\circ\psi \in \mathrm{Diff}_0(N)^{E=0}$.

(7) *We claim that* $\mathrm{Diff}_0(N)^{E=0}$ *is a non-trivial subgroup.* In view of the simplicity of $\mathrm{Diff}_0(N)$ mentioned in **5.1** this concludes the proof.

It remains to find a non-trivial diffeomorphism in $\mathrm{Diff}_0(N)^{E=0}$. The idea is to use compression waves. The basic case is this: take any non-decreasing smooth function $f : \mathbb{R} \to \mathbb{R}$ such that $f(x) \equiv 0$ if $x \ll 0$ and $f(x) \equiv 1$ if $x \gg 0$. Define

$$\varphi(t,x) = x + f(t - \lambda x)$$

where $\lambda < 1/\max(f')$. Note that

$$\varphi_x(t,x) = 1 - \lambda f'(t-\lambda x) > 0,$$

hence each map $\varphi(t,\quad)$ is a diffeomorphism of $\mathbb{R}$ and we have a path in the group of diffeomorphisms of $\mathbb{R}$. These maps are not the identity outside a compact set

however. In fact, $\varphi(x) = x + 1$ if $x \ll 0$ and $\varphi(x) = x$ if $x \gg 0$. As $t \to -\infty$, the map $\varphi(t, \quad)$ approaches the identity, while as $t \to +\infty$, the map approaches translation by 1. This path is a moving compression wave which pushes all points forward by a distance 1 as it passes. We calculate its energy between two times $t_0$ and $t_1$:

$$
\begin{aligned}
E_{t_0}^{t_1}(\varphi) &= \int_{t_0}^{t_1} \int_{\mathbb{R}} \varphi_t(t, \varphi(t, \quad)^{-1}(x))^2 dx\, dt = \int_{t_0}^{t_1} \int_{\mathbb{R}} \varphi_t(t,y)^2 \varphi_y(t,y) dy\, dt \\
&= \int_{t_0}^{t_1} \int_{\mathbb{R}} f'(z)^2 \cdot (1 - \lambda f'(z)) dz/\lambda\, dt \\
&\leq \frac{\max f'^2}{\lambda} \cdot (t_1 - t_0) \cdot \int_{\operatorname{supp}(f')} (1 - \lambda f'(z)) dz
\end{aligned}
$$

If we let $\lambda = 1 - \varepsilon$ and consider the specific $f$ given by the convolution

$$f(z) = \max(0, \min(1, z)) \star G_\varepsilon(z),$$

where $G_\varepsilon$ is a smoothing kernel supported on $[-\varepsilon, +\varepsilon]$, then the integral is bounded by $3\varepsilon$, hence

$$E_{t_0}^{t_1}(\varphi) \leq (t_1 - t_0) \tfrac{3\varepsilon}{1-\varepsilon}.$$

We next need to adapt this path so that it has compact support. To do this we have to start and stop the compression wave, which we do by giving it variable length. Let:

$$f_\varepsilon(z, a) = \max(0, \min(a, z)) \star (G_\varepsilon(z) G_\varepsilon(a)).$$

The starting wave can be defined by:

$$\varphi_\varepsilon(t, x) = x + f_\varepsilon(t - \lambda x, g(x)), \quad \lambda < 1, \quad g \text{ increasing}.$$

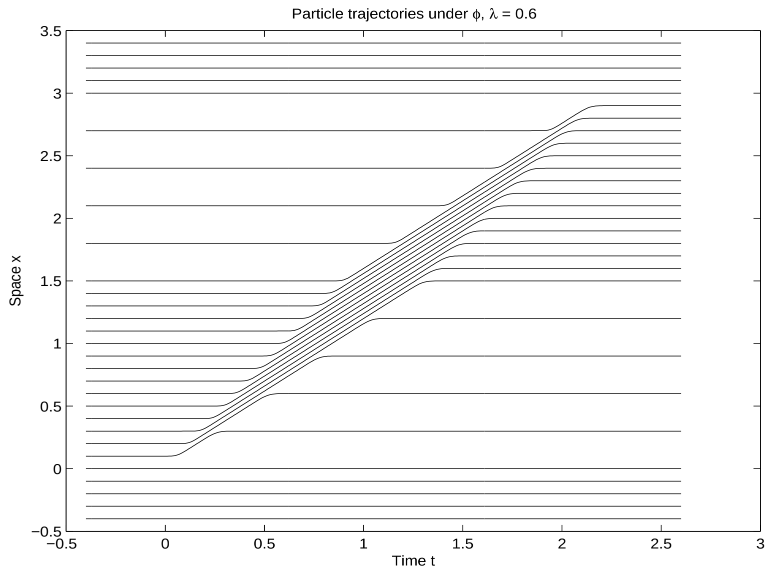
Note that the path of an individual particle $x$ hits the wave at $t = \lambda x - \varepsilon$ and leaves it at $t = \lambda x + g(x) + \varepsilon$, having moved forward to $x + g(x)$. Calculate the derivatives:

$$
\begin{aligned}
(f_\varepsilon)_z &= I_{0 \leq z \leq a} \star (G_\varepsilon(z) G_\varepsilon(a)) \in [0,1] \\
(f_\varepsilon)_a &= I_{0 \leq a \leq z} \star (G_\varepsilon(z) G_\varepsilon(a)) \in [0,1] \\
(\varphi_\varepsilon)_t &= (f_\varepsilon)_z (t - \lambda x, g(x)) \\
(\varphi_\varepsilon)_x &= 1 - \lambda (f_\varepsilon)_z (t - \lambda x, g(x)) + (f_\varepsilon)_a (t - \lambda x, g(x)) \cdot g'(x) > 0.
\end{aligned}
$$

This gives us:

$$
\begin{aligned}
E_{t_0}^{t_1}(\varphi) &= \int_{t_0}^{t_1} \int_{\mathbb{R}} (\varphi_\varepsilon)_t^2 (\varphi_\varepsilon)_x dx\, dt \\
&\leq \int_{t_0}^{t_1} \int_{\mathbb{R}} (f_\varepsilon)_z^2 (t - \lambda x, g(x)) \cdot (1 - \lambda (f_\varepsilon)_z (t - \lambda x, g(x))) dx\, dt \\
&\quad + \int_{t_0}^{t_1} \int_{\mathbb{R}} (f_\varepsilon)_z^2 (t - \lambda x, g(x)) \cdot (f_\varepsilon)_a (t - \lambda x, g(x)) g'(x) dx\, dt
\end{aligned}
$$

The first integral can be bounded as in the original discussion. The second integral is also small because the support of the $z$-derivative is $-\varepsilon \leq t - \lambda x \leq g(x) + \varepsilon$, while the support of the $a$-derivative is $-\varepsilon \leq g(x) \leq t - \lambda x + \varepsilon$, so together $|g(x) - (t - \lambda x)| \leq \varepsilon$.




Now define $x_1$ and $x_2$ by $g(x_1) + \lambda x_1 = t + \varepsilon$ and $g(x_0) + \lambda x_0 = t - \varepsilon$. Then the inner integral is bounded by

$$\int_{|g(x)+\lambda x - t| \le \varepsilon} g'(x)dx = g(x_1) - g(x_0) \le 2\varepsilon,$$

and the whole second term is bounded by $2\varepsilon(t_1 - t_0)$. Thus the length is $O(\varepsilon)$.

The end of the wave can be handled by playing the beginning backwards. If the distance that a point $x$ moves when the wave passes it is to be $g(x)$, so that the final diffeomorphism is $x \mapsto x + g(x)$ then let $b = \max(g)$ and use the above definition of $\varphi$ while $g' > 0$. The modification when $g' < 0$ (but $g' > -1$ in order for $x \mapsto x + g(x)$ to have positive derivative) is given by:

$$\varphi_\varepsilon(t, x) = x + f_\varepsilon(t - \lambda x - (1 - \lambda)(b - g(x)), g(x)).$$

A figure showing the trajectories $\varphi_\varepsilon(t, x)$ for sample values of $x$ is shown in the figure above.

It remains to show that $\mathrm{Diff}_0(N)^{E=0}$ is a nontrivial subgroup for an arbitrary Riemannian manifold. We choose a piece of a unit speed geodesic containing no conjugate points in $N$ and Fermi coordinates along this geodesic; so we can assume that we are in an open set in $\mathbb{R}^m$ which is a tube around a piece of the $u^1$-axis. Now we use a small bump function in the the slice orthogonal to the $u^1$-axis and multiply it with the construction from above for the coordinate $u^1$. Then it follows that we get a nontrivial diffeomorphism in $\mathrm{Diff}_0(N)^{E=0}$ again. □

**Remark.** Theorem **5.2** can possibly be proved directly without the help of the simplicity of $\mathrm{Diff}_0(N)$. For $N = \mathbb{R}$ one can use the method of **5.2**, (7) in the parameter space of a curve, and for general $N$ one can use a Morse function on $N$ to produce a special coordinate for applying the same method, as we did in the proof of theorem **3.2**.

**5.3**. **Geodesics and sectional curvature for $G^0$ on** $\mathrm{Diff}(N)$**.** According to Arnold [1], see [11], 3.3, for a right invariant weak Riemannian metric $G$ on an (possibly infinite dimensional) Lie group the geodesic equation and the curvature are given in terms of the adjoint operator (with respect to $G$, if it exists) of the Lie bracket by the following formulas:

$$\begin{aligned} u_t &= -\operatorname{ad}(u)^* u, \quad u = \varphi_t \circ \varphi^{-1} \\ G(\operatorname{ad}(X)^* Y, Z) &:= G(Y, \operatorname{ad}(X) Z) \\ 4G(R(X,Y)X,Y) &= 3G(\operatorname{ad}(X)Y, \operatorname{ad}(X)Y) - 2G(\operatorname{ad}(Y)^* X, \operatorname{ad}(X)Y) \\ &\quad - 2G(\operatorname{ad}(X)^* Y, \operatorname{ad}(Y)X) + 4G(\operatorname{ad}(X)^* X, \operatorname{ad}(Y)^* Y) \\ &\quad - G(\operatorname{ad}(X)^* Y + \operatorname{ad}(Y)^* X, \operatorname{ad}(X)^* Y + \operatorname{ad}(Y)^* X) \end{aligned}$$

In our case, for $\mathrm{Diff}_0(N)$, we have $\operatorname{ad}(X)Y = -[X,Y]$ (the bracket on the Lie algebra $\mathfrak{X}_c(N)$ of vector fields with compact support is the negative of the usual one), and:

$$\begin{aligned} G^0(X,Y) &= \int_N g(X,Y)\operatorname{vol}(g) \\ G^0(\operatorname{ad}(Y)^* X, Z) &= G^0(X, -[Y,Z]) = \int_N g(X, -\mathcal{L}_Y Z)\operatorname{vol}(g) \\ &= \int_N g\Big(\mathcal{L}_Y X + (g^{-1}\mathcal{L}_Y g)X + \operatorname{div}^g(Y)X, Z\Big)\operatorname{vol}(g) \\ \operatorname{ad}(Y)^* &= \mathcal{L}_Y + g^{-1}\mathcal{L}_Y(g) + \operatorname{div}^g(Y)\operatorname{Id}_T N = \mathcal{L}_Y + \beta(Y), \end{aligned}$$

where the tensor field $\beta(Y) = g^{-1}\mathcal{L}_Y(g) + \operatorname{div}^g(Y)\operatorname{Id} : TN \to TN$ is self adjoint with respect to $g$. Thus the geodesic equation is

$$u_t = -(g^{-1}\mathcal{L}_u(g))(u) - \operatorname{div}^g(u)u = -\beta(u)u, \qquad u = \varphi_t \circ \varphi^{-1}.$$

The main part of the sectional curvature is given by:

$$\begin{aligned} &4G(R(X,Y)X,Y) = \\ &= \int_N \Big(3\|[X,Y]\|_g^2 + 2g((\mathcal{L}_Y + \beta(Y))X, [X,Y]) + 2g((\mathcal{L}_X + \beta(X))Y, [Y,X]) \\ &\qquad + 4g(\beta(X)X, \beta(Y)Y) - \|\beta(X)Y + \beta(Y)X\|_g^2\Big)\operatorname{vol}(g) \\ &= \int_N \Big(-\|\beta(X)Y - \beta(Y)X + [X,Y]\|_g^2 - 4g([\beta(X),\beta(Y)]X, Y)\Big)\operatorname{vol}(g) \end{aligned}$$

So sectional curvature consists of a part which is visibly non-negative, and another part which is difficult to decompose further.

**5.4**. **Example: Burgers' equation.** For $(N,g) = (\mathbb{R}, \mathrm{can})$ or $(S^1, \mathrm{can})$ the geodesic equation is *Burgers' equation* [2], a completely integrable infinite dimensional system,

$$u_t = -3u_x\, u, \qquad u = \varphi_t \circ \varphi^{-1}$$

and we get $G^0(R(X,Y)X,Y) = -\int [X,Y]^2\, dx$ so that all sectional curvatures are non-negative.

**5.5**. **Example: $n$-dimensional analog of Burgers' equation.** For $(N,g) = (\mathbb{R}^n, \mathrm{can})$ or $((S^1)^n, \mathrm{can})$ we have:

$$(\mathrm{ad}(X)Y)^k = \sum_i ((\partial_i X^k)Y^i - X^i(\partial_i Y^k))$$
$$(\mathrm{ad}(X)^* Z)^k = \sum_i \Big((\partial_k X^i)Z^i + (\partial_i X^i)Z^k + X^i(\partial_i Z^k)\Big),$$

so that the geodesic equation is given by

$$\partial_t u^k = -(\mathrm{ad}(u)^\top u)^k = -\sum_i \Big((\partial_k u^i)u^i + (\partial_i u^i)u^k + u^i(\partial_i u^k)\Big),$$

called the basic Euler-Poincaré equation (EPDiff) in [6], the $n$-dimensional analog of Burgers' equation.

**5.6**. **Stronger metrics on** $\mathrm{Diff}_0(N)$**.** A very small strengthening of the weak Riemannian $H^0$-metric on $\mathrm{Diff}_0(N)$ makes it into a true metric. We define the stronger right invariant semi-Riemannian metric by the formula:

$$G^A_\varphi(h,k) = \int_N (g(X,Y) + A\,\mathrm{div}_g(X).\,\mathrm{div}_g(Y))\,\mathrm{vol}(g).$$

Then the following holds:

**5.7. Theorem.** *For any distinct diffeomorphisms $\varphi_0, \varphi_1$, the infimum of the lengths of all paths from $\varphi_0$ to $\varphi_1$ with respect to $G^A$ is positive.*

This implies that the metric $G^0$ induces positive geodesic distance on the subgroup of volume preserving diffeomorphism since it coincides there with the metric $G^A$.

**Proof.** Let $\psi_1 = \varphi_0 \circ \varphi_1^{-1}$. If $\varphi_0 \neq \varphi_1$, there are two functions $\rho$ and $f$ on $N$ with compact support such that:

$$\int_N \rho(y) f(\psi_1(y))\,\mathrm{vol}(g)(y) \neq \int_N \rho(y) f(y)\,\mathrm{vol}(g)(y).$$

Now consider any path $\varphi(t,y)$ between the two maps with derivative $u = \varphi_t \circ \varphi^{-1}$. Inverting the diffeomorphisms (or switching from a Lagrangian to an Eulerian point of view), let $\psi(t, \quad) = \varphi(0, \quad) \circ \varphi(t, \quad)^{-1}$. Then $\psi_t = -T\psi(u)$ and we have:

$$\begin{aligned}
&\int_N \rho(y) f(\psi_1(y))\,\mathrm{vol}(g)(y) - \int_N \rho(y) f(y)\,\mathrm{vol}(g)(y) = \\
&\quad = \int_0^1 \int_N \rho(y) \partial t f(\psi(t,y)\,\mathrm{vol}(g)(y) dt = \int_0^1 \int_N \rho(y)(df \circ \psi)(\psi_t(t,y))\,\mathrm{vol}(g)(y)\,dt \\
&\quad = \int_0^1 \int_N \rho(y)(Tf \circ \psi)(-T\psi(u(t,y)))\,\mathrm{vol}(g)(y) dt
\end{aligned}$$

But $\mathrm{div}((f \circ \psi) \cdot \rho u) = (f \circ \psi) \cdot \mathrm{div}(\rho u) + (Tf \circ \psi)(T\psi(\rho u))$. The integral of the left hand side is 0, hence:

$$\Big| \int_N \rho(y) f(\psi_1(y))\,\mathrm{vol}(g)(y) - \int_N \rho(y) f(y)\,\mathrm{vol}(g)(y) \Big|$$

$$= \Big| \int_0^1 \int_N (f \circ \psi) \operatorname{div}(\rho u) \operatorname{vol}(g) dt \Big|$$
$$\le \sup(|f|) \int_0^1 \sqrt{\int_N C_\rho \|u\|^2 + C'_\rho |\operatorname{div}(u)|^2 \operatorname{vol}(g)} dt$$

for constants $C_\rho, C'_\rho$ depending only on $\rho$. Clearly the right hand side is a lower bound for the length of any path from $\varphi_0$ to $\varphi_1$. $\square$

**5.8. Geodesics for $G^A$ on** $\operatorname{Diff}(\mathbb{R})$**.** See [3] and [12]. We consider the groups $\operatorname{Diff}_c(\mathbb{R})$ or $\operatorname{Diff}(S^1)$ with Lie algebras $\mathfrak{X}_c(\mathbb{R})$ or $\mathfrak{X}(S^1)$ with Lie bracket $\operatorname{ad}(X)Y = -[X,Y] = X'Y - XY'$. The $G^A$-metric equals the $H^1$-metric on $\mathfrak{X}_c(\mathbb{R})$, and we have:

$$\begin{aligned}
G^A(X,Y) &= \int_{\mathbb{R}} (XY + AX'Y')dx = \int_{\mathbb{R}} X(1 - A\partial_x^2)Y\,dx,\\
G^A(\operatorname{ad}(X)^*Y, Z) &= \int_{\mathbb{R}} (YX'Z - YXZ' + AY'(X'Z - XZ')')dx\\
&= \int_{\mathbb{R}} Z(1-\partial_x^2)(1-\partial_x^2)^{-1}(2YX' + Y'X - 2AY''X' - AY'''X)dx,\\
\operatorname{ad}(X)^*Y &= (1-\partial_x^2)^{-1}(2YX' + Y'X - 2AY''X' - AY'''X)\\
\operatorname{ad}(X)^* &= (1-\partial_x^2)^{-1}(2X' + X\partial_x)(1 - A\partial_x^2)
\end{aligned}$$

so that the geodesic equation in Eulerian representation $u = (\partial_t f) \circ f^{-1} \in \mathfrak{X}_c(\mathbb{R})$ or $\mathfrak{X}(S^1)$ is

$$\begin{aligned}
\partial_t u &= -\operatorname{ad}(u)^*u = -(1-\partial_x^2)^{-1}(3uu' - 2Au''u' - Au'''u), \text{ or}\\
u_t - u_{txx} &= Au_{xxx}.u + 2Au_{xx}.u_x - 3u_x.u,
\end{aligned}$$

which for $A = 1$ is the *Camassa-Holm equation* [3], another completely integrable infinite dimensional Hamiltonian system. Note that here geodesic distance is a well defined metric describing the topology.

## References


[1] V.I. Arnold. Sur la géometrie différentielle des groupes de Lie de dimension infinie et ses applications à l'hydrodynamique des fluides parfaits. *Ann. Inst. Fourier* 16 (1966), 319–361.

[2] J. Burgers. A mathematical model illustrating the theory of turbulence. *Adv. Appl. Mech.* 1 (1948), 171–199.

[3] R. Camassa and D. Holm. An integrable shallow water equation with peaked solutions. *Phys. Rev. Lett.* 71 (1993), 1661-1664.

[4] Vicente Cervera, Francisca Mascaro, and Peter W. Michor. The orbit structure of the action of the diffeomorphism group on the space of immersions. *Diff. Geom. Appl.*, 1 (1991), 391–401.

[5] D.B.A. Epstein. The simplicity of certain groups of homeomorphisms. *Compositio Math.* 22 (1970), 165–173.

[6] D. Holm, T. Ratnanather, A. Trouve and L. Younes. Soliton Dynamics in Computational Anatomy *NeuroImage*, Sept. 2004.

[7] Gerd Kainz. A metric on the manifold of immersions and its Riemannian curvature. *Monatsh. Math.*, 98 (1984), 211-217.

[8] Andreas Kriegl and Peter W. Michor. *The Convenient Setting for Global Analysis*. AMS, Providence, 1997. 'Surveys and Monographs 53'.

[9] John N. Mather. Commutators of diffeomorphisms I, II. *Comm. Math. Helv.* 49 (1974), 512–528; 50 (1975), 33–40.

[10] Peter W. Michor and David Mumford. *Riemannian geometries on spaces of plane curves.* arXiv:math.DG/0312384.
[11] Peter W. Michor and Tudor Ratiu. Geometry of the Virasoro-Bott group. *J. Lie Theory*, 8 (1998), 293–309. arXiv:math.DG/9801115
[12] G. Misiołek. A shallow water equation as a geodesic flow on the Bott-Virasoro group. *J. Geom. Phys.* 24 (1998), 203–208.
[13] Richard Palais. On the existence of slices for actions of non-compact Lie groups. *Ann. of Math.* 73 (1961), 295–323.
[14] William Thurston. Foliations and groups of diffeomorphisms. *Bull. AMS* 80 (1974), 304–307.

Peter W. Michor: Fakultät für Mathematik, Universität Wien, Nordbergstrasse 15, A-1090 Wien, Austria; *and:* Erwin Schrödinger Institut für Mathematische Physik, Boltzmanngasse 9, A-1090 Wien, Austria

*E-mail address*: Peter.Michor@univie.ac.at

David Mumford: Division of Applied Mathematics, Brown University, Box F, Providence, RI 02912, USA

*E-mail address*: David_Mumford@brown.edu